

Applications of Fuzzy Theories to Multi-Objective System Optimization

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Prepared for
Ames Research Center
University Consortium Interchange Number NCA2-223
January 1991



National Aeronautics and
Space Administration

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NOTATION

\vec{f}	vector of objective functions
f_i	i th objective function
F_i	i th normalized (standardized) objective function
F_{0ix}	inertia force on i th link in x-dirn.
F_{0iy}	inertia force on i th link in y-dirn.
g_j	j th inequality constraint
I_i	inertia of i th link
\bar{I}_i	inertia of i th counterweighted link
I_{ci}	inertia of i th counterweight
k	number of objective functions
L^i	lower bound on F_i
m	number of inequality constraints
m_i	scaling factor for the i th objective (f_i)
\bar{m}_i	mass of i th counterweighted link
m_{ci}	mass of i th counterweight
M	constant
r_{ci}	cg of i th counterweight
\bar{R}_i	cg of i th counterweighted link
S	feasible solution space, supercriterion
T_{0i}	inertia torque on i th link
U^i	upper bound on F_i
w_i	weighting factor for the i th objective
X	set of feasible design variables
X_0	starting design vector
X^*	optimum design vector
X_i^*	optimum design vector for i th objective
$[]^T$	transpose of $[]$
ϵ_i	structural error at i th design position
\cap	fuzzy intersection
μ_0	grade of membership of $()$
DM	decision maker (designer)
IGE	in ground effect
OEI	one engine inoperative
OGE	out of ground effect
SLS	sea level standard conditions

1. INTRODUCTION

Recent advances in engineering optimization have resulted in the development of techniques for handling problems involving large numbers of design variables and/or constraints (Schmit 1981, Vanderplaats 1982). Usually a scalar-valued objective function is optimized over a feasible design space and the result is often used as a guiding device in striving for the best possible system. However, there often exist several engineering design problems, which involve several, often conflicting, objectives to be considered by the designer.

The earliest work reporting the consideration of multiple objectives in mathematical programming appears to be that of Kuhn and Tucker (1950). The progress in the field of multicriteria optimization was summarized by Hwang and Masud (1979), Evans (1984), and Stadler (1984). The consideration of competing design objectives in mechanical systems using heuristic methods was discussed by Bartel and Marks (1974). The importance of game theory as a design tool has been emphasized by Rao and Hati (1980) and Vincent (1983). Since no unique solution, which would be optimum for all the individual objective functions, exists for a multiobjective optimization problem, the concept of Pareto-optimality has been used in most of the available methods. Several techniques for generating Pareto-optimal solutions are presented in this work.

In modeling most real world problems, a designer is often forced to state a problem in precise mathematical terms rather than in terms of the real world which is often imprecise in nature. The relationships and statements used for description may be imprecise not due to randomness but because of inherent fuzziness in the system. Fuzziness is a type of imprecision associated with fuzzy sets in which there is no sharp transition from membership to non-membership. Further, with increasing system complexity, one's ability to make precise and significant statements concerning a given system diminishes (Zadeh 1973). Consequently, the closer one examines a real-world problem, the fuzzier its description becomes. Fuzzy set theories can effectively model such domains in which the description of activities and observations are "fuzzy", in the sense that there are no sharply defined boundaries of the set of activities or observations to which the descriptions apply. These theories enable one to structure and describe activities which differ from each other vaguely, to formulate them in models, and to use these models for problem solving and decision making.

Fuzzy set theory was initiated by Zadeh in 1965. Since then for some ten years, the mathematics of the subject was developed but few applications resulted. During the last decade, these theories have been applied to various areas such as artificial intelligence, control, image processing, pattern recognition, robotics, psychology, etc. The first application of fuzzy theories to decision making processes was presented by Bellman and Zadeh (1970). This paper prescribed basic concepts and definitions associated with a decision making process in a fuzzy environment. Since then, these conceptual techniques have been employed to formulate and solve several mathematical programming problems.

Zimmermann has applied fuzzy optimization techniques to linear programming problems with single (1976) and multiple objectives (1978). An application of these theories to Preemptive and Archimedian versions of goal programming problems has been presented by Hannan (1981). Wang and Wang (1985) have used the method of level cut solutions for the fuzzy optimum design of structures. Rao has employed fuzzy optimization techniques for the design of

mechanical (1987a) and structural systems (1987b). An application of these techniques to multiobjective, multiple attribute decision making problems has been presented by Yager (1979). The concept of efficient and weakly efficient solutions in the context of fuzzy multiobjective problems has been discussed by Feng (1983) and Negotia (1981).

This work demonstrates the application and effectiveness of fuzzy theories in the formulation and solution of two types of helicopter design problems involving multiple objectives. The first problem deals with the determination of optimum flight parameters to accomplish a specified mission in the presence of three competing objectives. The second problem addresses the optimum design of the main rotor of a helicopter involving eight objective functions. A method of solving the resulting fuzzy multiobjective problem using nonlinear programming techniques is presented. Results obtained using fuzzy formulation are compared with those obtained using crisp optimization techniques. The outlined procedure should be useful in engineering design situations where uncertainty arises about the preciseness of permissible parameters, degree of credibility, and correctness of statements and judgements.

The fuzzy approach is also applied to the problem of integrated design of high speed planar mechanisms. The integrated formulation combines both the kinematic and dynamic synthesis aspects of mechanism design. The multiobjective optimization techniques presented in this work facilitate the design of a linkage to meet several kinematic and dynamic design criteria. The method can be used for motion, path, and function generation problems. The nonlinear programming formulation also permits the imposition of constraints to eliminate solutions which possess undesirable kinematic and motion characteristics. To model the vague and imprecise information in the problem formulation, the tools of fuzzy set theory have been used. A method of solving the resulting fuzzy multiobjective problem using mathematical programming techniques is presented. In addition, several nonlinear shapes for membership functions are considered to determine their impact on the overall design process. It has been observed that the final design is strongly influenced by the nature of designer's behavior with respect to fuzzy objectives and constraints.

2. MULTIPLE OBJECTIVE DECISION MAKING - CRISP APPROACH

2.1 PROBLEM STATEMENT

A general multiple objective nonlinear programming (NLP) problem is of the following form

$$\text{Minimize } \vec{f}(X)$$

subject to

$$X \in S = [X \mid X \in R^n, g_j(X) \leq 0] \quad (1)$$

where

$$X = [x_1, x_2, \dots, x_n]^T \quad (2)$$

$$\vec{f}(X) = [f_1(X), f_2(X), \dots, f_k(X)]^T. \quad (3)$$

For a single objective optimization problem, an optimum solution is defined as one that minimizes the objective function $f_i(X)$ subject to the constraints $g_j(X) \leq 0, j=1,2,\dots,m$.

Attempting to define a vector minimal point as one at which all components of the objective function vector \vec{f} are simultaneously minimized is not an adequate generalization since such "utopia" points are seldom attainable. Thus a new optimality concept, different from that used in scalar optimization, is necessary to find a solution to the vector optimization problem. The concept of a Pareto-optimal solution (Soland 1979, Steuer 1986) has been found to be useful in this context.

Definition 1: A feasible solution $X^* \in S$ is Pareto-optimal if there is no $\bar{X} \in S$ such that $f_i(\bar{X}) \leq f_i(X^*)$, $i=1,2,\dots,k$, and $f_{i_0}(\bar{X}) < f_{i_0}(X^*)$ for at least one $i_0 \in [1,\dots,k]$.

Alternately, a design vector X^* is Pareto-optimal if there exists no feasible vector \bar{X} which would decrease some objective function without causing a simultaneous increase in at least one other objective function. Unless a problem is convex, only a locally optimal solution can be guaranteed using standard mathematical programming techniques. Thus, the concept of Pareto-optimality needs to be defined for a nonconvex problem as

Definition 2: A solution $X^* \in S$ is said to be locally Pareto-optimal if and only if there exists a $\delta > 0$ such that X^* is Pareto-optimal in $S \cap N(X^*, \delta)$ where $N(X^*, \delta)$ denotes a neighborhood of X^* , i.e. the set $\{X \mid X \in S, X \in R^n, \|X - X^*\|_2 < \delta\}$.

The set of Pareto-optimal solutions usually consists of an infinite number of points and additional information is needed to order the Pareto-optimal set (Rosenthal 1985). This makes it possible to bring in additional considerations which are not included in the optimization model, thus making the multiobjective approach a flexible technique for most design problems. Several numerical techniques have been suggested for solving a vector optimization problem. Each method, in general, generates a different Pareto-optimal solution which reflects the decision maker's (DM's) preference structure.

2.2 SOLUTION TECHNIQUES

We now present some of the commonly used techniques (Rao 1984, Dhingra et al. 1990b) to solve the vector minimization problem given by Eq. (1). In order to have a common basis for comparison, and to avoid working with different objectives in different units, the objective functions $f_i(X)$ are transformed into new objective functions (F_i) constructed as follows

$$F_i(X) = m_i f_i(X) \quad i=1,2,\dots,k. \quad (4)$$

Here, the positive constant multipliers m_1, m_2, \dots, m_k are chosen so that

$$m_1 f_1(X_0) = m_2 f_2(X_0) = \dots = m_k f_k(X_0) = M \quad (5)$$

at any feasible starting vector X_0 . This scaling procedure ensures that all the objective functions are equal at a particular value of X_0 . Hereafter, it will be assumed that the k objective functions correspond to the k scaled objective functions given by Eq. (4). Further, it will be assumed that the nonlinear vector minimization problem given by Eq. (1) is nonconvex so that only locally Pareto-optimal solutions are guaranteed. The nonconvexity assumption holds for most practical design problems.

The next few subsections discuss some of the techniques which are used to generate Pareto-optimal solutions for the mathematical programming problem given by Eq. (1). Each of these techniques require additional information from the DM, and in general, generate a different Pareto-optimal solution.

2.2.1 Global Criterion Formulation

This method belongs to a category of multiple objective optimization techniques which require no articulation of preferences on part of the decision maker once the problem objectives and constraints have been defined. This entails that the DM be willing to accept whatever solution is obtained by minimizing some global criterion $F(X)$, for example, the sum of the squares of the relative deviations of the individual objective functions from the feasible ideal solutions. In other words, an optimum solution X^* is found by minimizing

$$F(X) = \sum_{i=1}^k \left[\frac{F_i(X) - F_i(X_i^*)}{F_i(X_i^*)} \right]^p \quad (6)$$

subject to

$$g_j(X) \leq 0 \quad j=1,2,\dots,m.$$

The value of p corresponds to the utility function of the DM and is usually taken as 2. The X_i^* is the feasible ideal solution corresponding to the i th objective function, and is obtained by minimizing $F_i(X)$ with respect to the constraint set $X \in S$. For $1 \leq p < \infty$, each solution obtained by solving Eq. (6) is Pareto-optimal. Compromise solutions with $p = \infty$ correspond to min-max criterion for which Pareto-optimality is not guaranteed.

2.2.2 Utility Function Formulation

In this approach, the vector minimization problem (Eq. (1)) is converted to

$$\text{Maximize } U(\vec{f}) \quad (7)$$

subject to

$$g_j(X) \leq 0 \quad j=1,2,\dots,m$$

where $U(\vec{f})$ is the utility function of multiple objective functions. The rationale for using $U(\vec{f})$ is that the DM has some utility associated with each of the k objective functions. A utility function U can have many forms (Farquhar 1977, Klein et al. 1985). The most common form assumes that the DM's utility function is additively separable with respect to all the objective functions. Thus, if $U_i(F_i)$ is the utility function corresponding to the objective function F_i , an overall utility function U is defined as

$$U(\vec{F}) = \sum_{i=1}^k U_i(F_i). \quad (8)$$

An optimum solution vector X^* is found by maximizing the total utility $U(\vec{F})$ (Eq. (8)) subject to the constraint set $g_j(X) \leq 0$. A special form of Eq. (8) which has been extensively used in multiobjective problems is given by

$$U = - \sum_{i=1}^k w_i F_i(X) \quad (9)$$

where w_i is a scalar weighting factor associated with the i th objective function and indicates its relative importance. This additively separable form of the utility function (Eq. (9)) is also commonly referred to as the weighting method, and serves as a sufficient condition for the calculation of Pareto-optimal solutions.

The main advantage of the utility function formulation is its simplicity. It is easier to assess k unidimensional utility functions (U_i 's) than to assess $U(\vec{F})$ directly. Similarly, it is easier to get w_i 's from the decision maker. The disadvantage of this approach are there are few cases where utility function is really additively separable, and w_i depend not only on the achievement level of F_i but also the achievement level of F_i relative to F_j , for $i \neq j$. Further, if the problem is nonconvex, this approach may miss all but a finite number of Pareto-optimal solutions.

2.2.3 Goal Programming Method

In goal programming, there are two basic models: the Archimedian model and the Preemptive model. The Archimedian model deals with generation of candidate solutions whose criterion vectors are closest, in a weighted L_p metric sense to the utopian set in the criterion space. The preemptive model, on the other hand, generates solutions whose criterion vectors are most closely related in a lexicographic sense, to points in the utopian set. The Archimedian version of goal programming is considered in this work.

In the simplest version of Archimedian goal programming, a designer sets goals and relative weights for each of the objective functions that he/she wishes to attain. An optimum solution X^* is then defined as the one that minimizes the weighted sum of the deviations from the set goals. Thus, the goal programming formulation of a multiobjective problem leads to

$$\text{Minimize} \quad \left[\sum_{j=1}^k w_j \left(d_j^+ + d_j^- \right)^p \right]^{1/p} \quad p \geq 1 \quad (10)$$

subject to

$$g_i(X) \leq 0 \quad i=1,2,\dots,m$$

$$F_j(X) - d_j^+ + d_j^- = b_j \quad j=1,2,\dots,k \quad (11)$$

$$d_j^+ \geq 0 \quad j=1,2,\dots,k \quad (12)$$

$$d_j^- \geq 0 \quad j=1,2,\dots,k \quad (13)$$

$$d_j^+ d_j^- = 0 \quad j=1,2,\dots,k \quad (14)$$

where b_j are the goals set by the designer for the j th objective function, and d_j^+ and d_j^- are the under- and overachievement from the target goals for the j th objective function. The value of p is based on a utility function chosen by the designer. If the goals b_j are set equal to F_j^* obtained by minimizing individual objective functions F_j , it is not possible to obtain an overachievement of the goals b_j 's. Consequently, the d_j^- need not be defined. Thus the goal programming formulation given by Eqs. (10-14) reduces to

$$\text{Minimize } \left[\sum_{j=1}^k w_j \left[d_j^+ \right]^p \right]^{1/p} \quad p \geq 1 \quad (15)$$

subject to

$$\begin{aligned} g_i(X) &\leq 0 \quad i=1,2,\dots,m \\ d_j^+ &\geq 0 \quad j=1,2,\dots,k \\ d_j^+ &= F_j(X) - F_j^*(X) \quad j=1,2,\dots,k. \end{aligned} \quad (16)$$

The goal constraints in the above formulation are soft constraints in the sense that they do not restrict the original feasible region S . In effect, they augment the feasible region by casting S into higher dimensional space, thereby creating the augmented goal programming feasible region. However, if the goal vector is not chosen properly, there is no guarantee that the goal programming formulation will terminate at a Pareto-optimal solution.

2.2.4 Goal Attainment Method

Goal attainment formulation requires setting up goals b_1, b_2, \dots, b_k and weights w_1, w_2, \dots, w_k for the objective functions F_1, F_2, \dots, F_k respectively. The weights w_i relate the relative under- or overattainment of the desired goals (b_i). The following problem is solved to determine the optimal solution X^*

$$\text{Minimize } z \quad (17)$$

subject to

$$\begin{aligned} g_j(X) &\leq 0 \quad j=1,2,\dots,m \\ F_i(X) - w_i z &\leq b_i \quad i=1,2,\dots,k \end{aligned} \quad (18)$$

$$w_i \geq 0 \quad i=1,2,\dots,k \quad (19)$$

where z is a scalar variable unrestricted in sign. The weights w_i are normalized so that

$$\sum_{i=1}^k w_i = 1. \quad (20)$$

In the case of the underattainment of the desired goals, a smaller weighting coefficient is associated with the more important objective functions. For an overattainment of the desired goals, a smaller weighting coefficient is associated with the less important objective functions. The optimum solution obtained using the goal attainment formulation is fairly sensitive to the goal vector (\vec{b}) and the weighting vector (\vec{w}) given by the DM. Depending upon the prescribed values of the goal vector, it is possible that the weighting vector \vec{w} does not dictate the optimum solution at all. Instead, the optimum solution X^* is determined by the nearest nondominated solution point from \vec{b} . This may require that \vec{w} be varied parametrically to generate the entire set of Pareto-optimal solutions. Further, if the goal vector is not chosen properly, there is no guarantee that the goal attainment formulation will terminate at a Pareto-optimal solution.

2.2.5 Bounded Objective Function Formulation

In this method, the minimum and maximum acceptable achievement levels for each objective function F_i are specified by the DM as L^i and U^i respectively. Then, an optimum solution X^* is found by solving the following problem

$$\text{Minimize } F_r(X) \quad (21)$$

subject to

$$\begin{aligned} g_j &\leq 0 \quad j=1,2,\dots,m \\ L^i &\leq F_i(X) \leq U^i \quad i=1,2,\dots,k; \quad i \neq r. \end{aligned} \quad (22)$$

This method, also referred to as ϵ -constraint method, can be shown to lead to weak Pareto-optimal solutions. However, if the optimal solution to the above problem is unique, then the resulting solution is Pareto-optimal. Further, by systematically varying L^i and U^i , the bounded objective formulation can generate the entire set of Pareto-optimal solutions for even nonconvex problems.

A difficulty with this method is to prescribe values for L^i and U^i prior to any preliminary solution. Since the designer has to specify these values in an information void, this may result in the mathematical programming problem given by Eqs. (21-22) into a problem with inconsistent constraints. Another question which needs to be addressed with this approach is which objective should be used for $F_r(X)$.

2.2.6 Lexicographic Method

In the lexicographic method, the objectives are ranked in order of importance by the designer. An optimum solution X^* is obtained by minimizing the objective functions, starting with the most important one and proceeding according to the order of importance of the objectives. The rationale for this method is that individuals tend to make decisions in this manner.

Let the subscripts of the objectives denote not only the objective function number, but also the priority of the objective. The solution procedure is given as follows

- 1) Starting with X_0 , minimize $F_1(X)$ subject to the constraint set $g_j(X) \leq 0$. Let the resulting optimum solution be denoted as X_1^* and F_1^* .
- 2) Starting from X_1^* , minimize $F_2(X)$ subject to the constraint set $g_j(X) \leq 0$, and an additional constraint of the form $0.95 F_1^* \leq F_1(X) \leq 1.05 F_1^*$. Let the resulting solution be X_2^* , and $F_2^* = F_2(X_2^*)$.
- 3) Proceeding as outlined in step (2), at the i th stage the resulting problem is given as: Starting from X_{i-1}^* , minimize $F_i(X)$ subject to the constraint set $g_j(X) \leq 0$, and $i-1$ additional constraint of the form $0.95 F_j^* \leq F_j(X) \leq 1.05 F_j^*$, $j=1,2,\dots,i-1$.

For a problem involving k criteria, there are a total of $k!$ ways in which the objective functions can be ranked by the DM. Since the solution obtained using the lexicographic method is fairly sensitive to the ranking of the objectives given by the DM, one should exercise caution in applying this method when some objective functions are of nearly equal importance.

2.2.7 Game Theory Approach

In the cooperative version of game theory (Rao and Hati 1980), a multiobjective optimization problem is viewed as a game problem involving several players, one corresponding to each of the objective functions. The system is assumed to be under the control of these intelligent adversaries, each willing to compromise his/her own objective in order to improve the overall solution. The basic approach is summarized as follows

i) Using X_0 as a starting point, solve k single objective optimization problems given by

$$\text{Minimize } F_i(X)$$

subject to

$$g_j(X) \leq 0 \quad j=1,2,\dots,m \quad (23)$$

Let the optimum solutions be X_i^* , $i=1,2,\dots,k$.

ii) Construct a supercriterion or bargaining model S as

$$S = \prod_{i=1}^k \left[F_{iu} - F_i(X_w^*) \right] \quad (24)$$

where

$$F_{iu} = \max \left[F_i(X_j^*) \right] \quad i, j=1,2,\dots,k \quad (25)$$

and X_w^* represents the Pareto-optimal solution obtained by solving the following problem

$$\text{Minimize } F_w(w, X) = \sum_{i=1}^k w_i F_i(X) \quad (26)$$

subject to

$$g_j(X) \leq 0 \quad j=1,2,\dots,m$$

$$\sum_{i=1}^k w_i = 1 \quad (27)$$

$$w_i \geq 0 \quad i=1,2,\dots,k. \quad (28)$$

iii) Maximize the supercriterion and the find the optimal convex combination \vec{w}^* of the objective functions and the corresponding optimal solution to the problem, i.e. $X = X_w^*$. The game theory approach as presented above not only yields a Pareto-optimal solution, but also results in an optimum set of relative weights for the k objective functions.

3. MULTIPLE OBJECTIVE DECISION MAKING - FUZZY APPROACH

3.1 BASIC CONCEPTS

Traditional schemes for design optimization assume that all the design data are known precisely, that the constraints delimit a well defined set of feasible decisions, and that the objective function is well defined and is easy to formulate. An optimal decision is that combination of decision variables X^* which results in the "highest degree of satisfaction" for the objective function $f(X)$.

For a problem involving uncertainty and fuzziness in the design input data, this notion of optimization needs to be modified. The objective function and the constraints constitute a class of alternatives whose boundaries are not well defined. To deal with this imprecision quantitatively, the tools of fuzzy set theory can be used. The fuzzy objective function and the fuzzy constraints are characterized by their membership functions. Since the overall optimization process requires a simultaneous satisfaction of the objective function and the constraints, a decision or selection of a set of design variables is made by assuming that the constraints are independent (i.e. the membership function for constraint g_i is independent of membership function of constraint g_j , for $i \neq j$) and the logical *and* (min) operator corresponds to an intersection. This definition of a decision as the intersection of goals and constraints reflects the interpretation of *and* in the hard (min) sense. The logical *and* does not allow any compensation (tradeoff) at all, that is, an element of the intersection of two fuzzy sets cannot compensate a low membership value of one of the intersected sets by a higher membership value of the second one. However, the min operator is most frequently employed in fuzzy optimization problems (Zimmermann 1985) and has been used in the present work.

Consider a crisp nonlinear mathematical programming problem of the form

$$\text{Minimize } f(X)$$

subject to

$$g_j(X) \leq b_j, \quad j=1,2,\dots,m \quad (29)$$

where

$$X = (x_1, x_2, \dots, x_n)^T. \quad (30)$$

The fuzzy analogue of the crisp nonlinear programming problem (29) can be stated as

Find X such that

$$f(X) \in \bar{F} \quad (31)$$

$$g_j(X) \in \bar{G}_j, \quad j=1,2,\dots,m \quad (32)$$

where \bar{F}, \bar{G}_j denote the allowable tolerance interval for the fuzzy goal (f) and the fuzzy constraint functions (g_j). The bar over a symbol indicates that the expression or variable contains fuzzy information. The fuzzy constraint $g_j \in \bar{G}_j$ indicates that g_j is a member of \bar{G}_j such that $\mu_{\bar{G}_j}(g_j(X)) > 0$, where $\mu_{\bar{G}_j}$ is the membership function for the fuzzy set \bar{G}_j . A fuzzy feasible region is defined by considering all the fuzzy constraints as

$$\bar{R}(X) = \bigcap_{j=1}^m \bar{G}_j(X). \quad (33)$$

This gives the overall degree of satisfaction of design vector X with respect to all the fuzzy constraints. A design vector X is considered feasible provided $\mu_{\bar{R}}(X) > 0$. The differences in the membership degrees of two vectors X_1 and X_2 imply nothing but variations in the degree of satisfaction of X_1 and X_2 with respect to the constraint set. A fuzzy decision is now defined as the confluence of the fuzzy goal (\bar{F}) and the fuzzy constraints ($\bar{G}_1, \bar{G}_2, \dots, \bar{G}_m$) as:

$$\bar{D}(X) = \bar{F} \cap \bar{G}_1 \cap \bar{G}_2 \cap \dots \cap \bar{G}_m \quad (34)$$

and in terms of membership values as

$$\mu_{\bar{D}}(X) = \left[\mu_{\bar{F}}(X) \right] \cap \left[\bigcap_{j=1}^m \mu_{\bar{G}_j}(g_j(X)) \right]. \quad (35)$$

Because of the symmetry of this aggregation procedure with respect to fuzzy goals and fuzzy constraints, there is no longer any distinction between the goals and the constraints of a decision process. A sufficient condition for a unique maximum is that \bar{D} be a strongly convex set, namely, \bar{D} is convex with a unimodal membership function. An optimum solution X^* is one at which the membership function of \bar{D} attains its maximum, i.e.

$$\mu_{\bar{D}}(X^*) = \max \mu_{\bar{D}}(X), X \in \bar{D} \quad (36)$$

where

$$\mu_{\bar{D}}(X) = \min_j \left[\mu_{\bar{F}}(X), \mu_{\bar{G}_j}(g_j(X)) \right]. \quad (37)$$

3.2 SOLUTION STRATEGY FOR MULTIPLE OBJECTIVES

Consider a multiple objective optimization problem with k fuzzy goals f_1, f_2, \dots, f_k represented by fuzzy sets \bar{F}_i , $i=1, \dots, k$ and m fuzzy constraints g_1, g_2, \dots, g_m represented by fuzzy sets \bar{G}_j , $j=1, \dots, m$. By generalizing the analogy from the single objective function case, the resulting fuzzy decision is given as

$$\bar{D} = \bar{F}_1 \cap \bar{F}_2 \cap \dots \cap \bar{F}_k \cap \bar{G}_1 \cap \bar{G}_2 \cap \dots \cap \bar{G}_m. \quad (38)$$

In terms of corresponding membership values for the fuzzy goals and the fuzzy constraints, the resulting decision is

$$\mu_{\bar{D}}(X) = \left[\bigcap_{i=1}^k \mu_{\bar{F}_i}(X) \right] \cap \left[\bigcap_{j=1}^m \mu_{\bar{G}_j}(g_j(X)) \right] \quad (39)$$

or

$$\mu_{\bar{D}}(X) = \min_{i,j} \left[\mu_{\bar{F}_i}(X), \mu_{\bar{G}_j}(X) \right]. \quad (40)$$

An optimum solution X^* is one at which the membership function of the resulting decision \bar{D} is maximum, i.e.

$$\mu_{\bar{D}}(X^*) = \max \mu_{\bar{D}}(X), X \in \bar{D} \quad (41)$$

where $\mu_{\bar{D}}(X)$ is given by Eq. (40).

The shape of the membership functions such as a linear, concave, or convex function, for various objectives and constraints, can affect the optimum solution significantly. A linear approximation has been most commonly used because of simplicity and expediency. But other shapes for membership functions such as a concave or a convex function offer potential benefits in terms of realism. In the present work, several possible common shapes for the membership function of the various fuzzy goals are chosen consistent with varying perceptions of the decision maker. These shapes are discussed in detail in section 3.3 on "Nonlinear membership functions".

3.2.1 Computational Procedure

An efficient solution of the fuzzy multiobjective problem given by Eq. (41) is determined by (i) finding the solutions of the individual single objective optimization problems, (ii) determining the best and worst values for each of the objective functions, (iii) using these values as the boundaries of the fuzzy ranges for the fuzzy objective functions in the corresponding optimization problem, and (iv) solving the resulting fuzzy optimization problem.

A linear membership function of a fuzzy objective function, for example, is constructed as:

$$\mu_{\bar{F}_i}(X) = \begin{cases} 0, & \text{if } f_i(X) \geq f_i^{\max} \\ \frac{-f_i(X) + f_i^{\max}}{f_i^{\max} - f_i^{\min}} & \text{if } f_i^{\min} < f_i(X) < f_i^{\max}, i=1,2,\dots,k \\ 1, & \text{if } f_i(X) \leq f_i^{\min} \end{cases} \quad (42)$$

where $f_i^{\min} = \min_j f_i(X_j^*)$ and $f_i^{\max} = \max_j f_i(X_j^*)$, and X_j^* is the optimum design vector of the j th objective function. A linear membership function models a decision maker's constant marginally increasing (or decreasing) membership value over the parameter range of interest and is defined by fixing upper and lower levels of design parameter acceptability. When the fuzzy constraints are stated as

$$g_j(X) \leq b_j + d_j, \quad j=1,2,\dots,m \quad (43)$$

where d_j denotes the distance by which the boundary of the j th constraint is moved, the linear membership function for the j th constraint is constructed as:

$$\mu_{\bar{G}_j}(X) = \begin{cases} 0, & \text{if } g_j(X) \geq b_j + d_j \\ 1 - \left[\frac{g_j(X) - b_j}{d_j} \right] & \text{if } b_j < g_j(X) < b_j + d_j, \quad j=1,2,\dots,m \\ 1, & \text{if } g_j(X) \leq b_j. \end{cases} \quad (44)$$

Once the membership functions of the fuzzy objectives and the fuzzy constraints, i.e. $\mu_{\bar{F}_i}$ and $\mu_{\bar{G}_j}$ are known, the fuzzy optimization problem (Eq. (41)) can be posed as an equivalent crisp optimization problem as follows:

Find X and λ which

$$\text{Maximize } \lambda \quad (45)$$

subject to

$$\lambda \leq \mu_{\bar{F}_i}(X), \quad i=1,2,\dots,k \quad (46)$$

$$\lambda \leq \mu_{\bar{G}_j}(X), \quad j=1,2,\dots,m. \quad (47)$$

This problem can be solved using standard single objective nonlinear programming techniques.

3.3 NONLINEAR MEMBERSHIP FUNCTIONS

One of the major assumptions in solving fuzzy mathematical programming problems in the literature involves the use of linear membership functions for all fuzzy sets involved in a decision making process. A linear approximation is most commonly used because of its simplicity and is defined by fixing two points, the upper and lower levels of acceptability (Zimmermann 1976, Rao 1987a,b). If fuzzy set theory is to be considered a purely formal theory, such an assumption is acceptable, even though some kind of formal justification of this assumption would be desirable. If, however, fuzzy set theory is used to model real decision making processes, and an assertion is made that the resulting models are true models of reality, then some kind of empirical justification for this assumption is necessary. In view of this, several other (nonlinear) shapes for membership functions, such as concave or convex shaped membership functions are analyzed to determine their impact on the overall design process. The marginal rate of increase (or decrease) of membership values as a function of design parameter values is not constant for these nonlinear membership functions, as is the case with linear membership functions. These nonlinear shapes offer potential benefits in terms of realism and are chosen consistent with varying perceptions of the decision maker (designer).

Several different shapes for the (monotonically decreasing) membership functions corresponding to the fuzzy objective functions are presented, and later examined to determine their impact on overall design process. These shapes correspond to what we define as positive (convex), negative (concave), or zero (linear) value of the coefficient of membership satiation, $m(X)$ which is defined as follows (Dhingra et al. 1990c)

$$m(X) = \mu''(X), \quad (48)$$

where $\mu''(X)$ is the second derivative of the membership function. This definition is analogous to the Arrow-Pratt measure of risk aversion and the Dyer-Sarin (1982) measure of value satiation used in decision analysis for characterizing utility and measurable value functions respectively (Keeney and Raiffa 1976, Keelin 1981). It may be noted that this definition of $m(X)$ does not include $\mu'(X)$ because a linear transformation of membership functions is not possible, which is the case with utility or value functions. A positive value of $m(X)$ corresponds to increasing marginal membership values at a given value of X (convex functions). Similarly a negative value of $m(X)$ corresponds to a decreasing marginal membership values (concave functions), and $m(X)=0$ is equivalent to constant marginal membership values (linear functions). Second order effects which determine whether $m(X)$ is increasing, constant or decreasing over the parameter range of interest, while retaining its sign, are also considered. The sine and exponential ($k > 0$) functions model increasing and decreasing values of $m(X)$ over the range of definition ($m(X) > 0$). The logarithmic, quadratic, and exponential ($k < 0$) functions are used to model increasing, constant and decreasing values of $m(X)$ when the membership satiation coefficient is negative. While the satiation coefficient retains its sign for these five functions, the sign of $m(X)$ changes over the range of definition for a hyperbolic function. The membership function of a fuzzy goal can also be viewed as a kind of utility function representing the degree of satisfaction or acceptance. Some of the nonlinear shapes which we have considered are shown in Fig. 1 and are discussed below.

In the following five subsections dealing with different membership functions, z corresponds to a particular value of the fuzzy objective function (\bar{Z}) and z_{\min} and z_{\max} are the fuzzy lower and upper bounds of the fuzzy objective function.

3.3.1 Exponential

An exponential membership function is defined as

$$\mu_{\bar{Z}} = \begin{cases} 1 & \text{if } z \leq z_{\min} \\ 0 & \text{if } z \geq z_{\max} \\ \frac{e^{-k\delta} - e^{-k}}{1 - e^{-k}} & \text{otherwise} \end{cases} \quad (49)$$

where

$$\delta = \frac{z - z_{\min}}{z_{\max} - z_{\min}} \quad (50)$$

and k is a parameter prescribed by the decision maker. When $k > 0$, $\mu_{\bar{Z}}$ is convex and consequently models an increasing marginal rate of membership values. While $m(X)$ is positive, its value decreases over the entire range of interest. A negative value of $m(X)$ can also be modeled using the above function for the case when $k < 0$. Here again, the magnitude of $m(X)$ is decreasing over the range of definition.

3.3.2 Hyperbolic Function

The hyperbolic function is convex over a part of the objective function values and is concave over the remaining part. The rationale for such a shape (Friedman and Savage 1952) in our problem context is as follows: When the decision maker is worse off with respect to a goal, the decision maker tends to have a higher marginal rate of satisfaction with respect to that goal. A convex shape captures that behavior in the membership function. On the other hand, when one is better off with respect to a goal, one tends to have a smaller marginal rate of satisfaction. Such behavior is modeled using the concave portion of the membership function. The complete function is as follows:

$$\mu_{\bar{z}}(X) = 0.5 - 0.5 \tanh [(z - \bar{z}) \delta] \quad (51)$$

$$\delta = \frac{6}{z_{\max} - z_{\min}}. \quad (52)$$

The above function has a membership value of 0.5 when $z = z_{\text{avg}} = 0.5(z_{\min} + z_{\max})$, and is symmetric with respect to the point z_{avg} . The decision maker's $m(X)$ is positive and decreasing from $[z_{\min}, z_{\text{avg}}]$, and is negative and increasing from $[z_{\text{avg}}, z_{\max}]$, with z_{avg} being the point of inflection.

3.3.3 Quadratic Function

A quadratic function is used to model a negative, but constant value of $m(X)$ on part of the decision maker. The function is expressed as

$$a z^2 + b z + c = \mu_{\bar{z}}. \quad (53)$$

Assuming that

$$\mu_{\bar{z}} = \begin{cases} 1 & \text{if } z \leq z_{\min} \\ 0 & \text{if } z \geq z_{\max} \\ 0.5 & \text{if } z = z_{\text{avg}} \end{cases} \quad (54)$$

the values of a , b , and c can be determined by solving the equations:

$$a z_{\min}^2 + b z_{\min} + c = 1.0 \quad (55)$$

$$a z_{\max}^2 + b z_{\max} + c = 0.0 \quad (56)$$

$$a z_{\text{avg}}^2 + b z_{\text{avg}} + c = 0.5. \quad (57)$$

If z_{avg} is taken to be $0.5(z_{\min} + z_{\max})$, the quadratic form given by Eq. (53) degenerates to a linear form as a becomes equal to zero.

3.3.4 Logarithmic Function

A logarithmic function is also used to model decreasing marginal rates of membership values. The function is given as follows

$$\mu_{\bar{z}} = \begin{cases} 1 & \text{if } z \leq z_{\min} \\ 0 & \text{if } z \geq z_{\max} \\ a + \log(c - z) & \text{otherwise.} \end{cases} \quad (58)$$

This concave function is characterized by a negative value of membership satiation coefficient over the entire range of definition. However, the value of $m(X)$ is increasing over the parameter range of interest.

3.3.5 Sine Function

A sine function is used to model positive and increasing of $m(X)$ on part of the decision maker. This function is expressed as

$$\mu_{\bar{z}} = \begin{cases} 1 & \text{if } z \leq z_{\min} \\ 0 & \text{if } z \geq z_{\max} \\ 1 - \sin \frac{\pi}{2} \delta & \text{otherwise.} \end{cases} \quad (59)$$

where δ is given by Eq. (50).

3.4 FUZZY GAME THEORY APPROACH

A variety of techniques for multiobjective optimization have been considered in section 2.2. As will be seen later, each of these techniques, in general, generates a different solution. This is due to the fact that each formulation has a different underlying preference structure. Game theoretic class of methods for multiobjective optimization yield a unique solution which is Pareto-optimal, require minimal amount of subjective information from the DM, require no interpersonal comparison of utilities, are independent of positive linear transformations of f_i 's, and ensure that at the final solution all the objectives are acceptable.

In game theory, a multiobjective engineering design problem is viewed as a game where each player corresponds to an objective function. These players are competing with each other to improve their overall situation subject to a limited supply of resources. Two theories have been used to abstract the conflict of interest situation between the players; the non-cooperative model based on the concept of Nash equilibrium, and the cooperative theory based on the concept of a Pareto-optimal solution.

In the cooperative model, each player is considered a part of a team who is willing to compromise his/her own payoff in order to improve the situation as a whole. A cooperative game proceeds with the intent that the team wants to allocate the resources so that all the players are as better off as possible. The team must then decide as to how the resources should be allocated such that an improvement in the payoff of one player does not result in an unacceptable loss for another player.

It is possible to combine the positive aspects of game theoretic and fuzzy formulations to yield superior and more robust methodologies for multiple objective decision making. Such a methodology requires the introduction of new operators which are different from the ones used in fuzzy formulation in section 3.2. A completely general formulation involves consideration of design problems which have partly crisp and partly fuzzy objectives, as well as partly crisp and partly fuzzy constraints. Such a formulation is a subject of potential future research.

4. APPLICATIONS

The effectiveness of multiple objective optimization techniques for engineering design problems is now demonstrated via an application to three design problems. The first problem deals with the determination of optimum flight parameters, for a helicopter, to accomplish a specified mission in the presence of three competing design objectives. The second problem addresses the optimum design of main rotor blades of a helicopter to accomplish a specified mission in the presence of eight different objective functions. Design example three presents a novel approach for the design of high speed mechanisms where both kinematic and dynamic criteria are addressed simultaneously.

4.1 FLIGHT TRAJECTORY OPTIMIZATION

4.1.1 Introduction

The problem of flight trajectory optimization has not received much attention in the literature. Earlier attempts at this problem employed the principles of optimal control theory. Recently, Shruster and Carpas (1983) have used optimal design techniques to achieve maximum range for an unpowered gliding flight. The design variables for the seven segment trajectory used in the analysis were the angle of attack for each leg of the mission. Jenkinson and Simos (1985) have demonstrated the application of optimal design techniques to short haul, fixed wing routes. The design variables used in their formulation are the indicated air speed, throttle setting, and the propeller setting. A comprehensive review of the application of optimal design techniques to various aspects of helicopter design problems can be found in Ashley (1982) and Miura (1985).

4.1.2 Problem Formulation

Flight profile optimization addresses the need for determining optimum flight parameters to accomplish a specified mission for a given payload. The mission planning task is to select, prior to the flight, the altitude-speed profile and the initial fuel load for the mission. The objectives, for example, can be to minimize fuel cost, minimize flight time, minimize total cost, maximize range, or maximize payload, etc. Earlier attempts at solving this type of problem have employed the principles of optimal control theory (Schmitz 1971). However, if a mission can be discretized into a finite number of segments, and if the flight conditions remain constant over each segment, the flight trajectory optimization problem can be formulated as a standard mathematical programming problem (Bennett 1985). This approach has been adopted in the present work.

For any specified mission, the flight profile optimization program (FPOP) generates an initial mission description consisting of ten segments. This initial mission approximation assumes that the fuel tanks are full at takeoff. The power required, and the fuel flow necessary for each segment are computed based on atmospheric properties such as air density, temperature, and wind speed. At the end of each mission segment, flight parameters such as helicopter altitude, fuel consumed, flight time, gross weight, distance traveled, etc. are computed by FPOP. The performance characteristics of the helicopter are derived from actual test data and are expressed as follows: i) The power coefficient (C_p) is expressed as a continuous function of advance ratio (μ) for discrete values of thrust coefficient (C_t) using a seventh order polynomial representation; ii) The engine fuel flow is expressed as a function of shaft horsepower for discrete values of density altitude using a seventh order polynomial representation; iii) A seventh order polynomial is used to express hover power coefficient as a continuous function of thrust coefficient for hover OGE, and hover IGE.

The flight parameters such as the initial fuel load, indicated air speed at the beginning of each of the ten segments, and the rate of climb at the beginning of the first nine segments are varied during the design procedure. The rate of climb or descent for the fifth segment of a two way mission, and the rate of descent for the tenth segment are not true independent variables. The values of these parameters are computed based on the altitude values at the beginning of the fifth (or tenth) segment and the destination altitude. From the initial profile determined by FPOP, the NLP algorithm iterates the flight parameters until an optimum profile satisfying the imposed constraints is determined.

The design variables for this problem are i) the initial fuel load, ii) indicated air speed at the beginning of each of the ten segments, and iii) the rate of climb (or descent) at the beginning of the first nine segments for a one way mission; or rate of climb (or descent) for segments 1,2,3,4,6,7,8,9 for a two way mission. The following behavior constraints are considered in the problem formulation

1. Horsepower required for each segment \leq Horsepower available
2. Indicated air speed $\leq V_{NE}$ for each segment
3. Altitude for each segment \leq Maximum altitude
4. Takeoff weight \leq Maximum takeoff gross weight; or
Takeoff weight \leq Maximum weight for hover OGE; or
Takeoff weight \leq Maximum weight for hover IGE
5. Fuel required \leq Fuel available

6. Error in terminal altitude is within the prescribed limits.

The side constraints (on design variables) include the following

1. Rate of climb for each segment \leq Maximum rate of climb
2. Rate of descent for each segment \leq Maximum rate of descent.

Three objective functions, namely, the minimization of fuel cost, flight time, and total cost are considered with prescribed values for payload and range.

4.1.3 Numerical Results - Crisp Formulation

For this design example dealing with flight trajectory optimization, the mission requirements are specified in Table 1. A total of three objective functions, namely, minimization of fuel cost, flight time, and total cost are considered. The optimum mission parameters (design variables) obtained using single and various multiple objective optimization techniques are presented in Tables 2-4. The values of objective functions corresponding to these optimum mission parameters are given in Table 5. It may be noted from Table 5 that the three objectives considered are conflicting in nature. A minimum fuel cost design results in poor values for the flight time and total mission cost. Attempting to achieve a design with low flight time and/or total cost results in a high rate of fuel consumption which in turn leads to a high fuel cost. Single objective optimization techniques are unable to overcome this difficulty and yield a solution which is characterized by a superior performance on one objective function and a (generally) poor performance with respect to the remaining objective functions. Multiobjective optimization techniques presented in this work are able to achieve a compromise by permitting a tradeoff between the conflicting pairs of objectives. The resulting solutions exhibit good performance with respect to all the objective functions. The optimum flight paths obtained using single and some multiple objective optimization techniques are presented in Figs. 2-4.

4.1.4 Numerical Results - Fuzzy Formulation

The results from single objective optimizations yield a $k \times k$ matrix $[M]$ defined as follows (Dhingra et al. 1988, 1990a):

$$[M] = \begin{bmatrix} f_1(X_1^*) & f_2(X_1^*) & \dots & f_k(X_1^*) \\ f_1(X_2^*) & f_2(X_2^*) & \dots & f_k(X_2^*) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(X_k^*) & f_2(X_k^*) & \dots & f_k(X_k^*) \end{bmatrix} = \begin{bmatrix} 383.37 & 2.7485 & 1372.83 \\ 427.39 & 2.3675 & 1279.70 \\ 423.13 & 2.3592 & 1272.43 \end{bmatrix} \quad (60)$$

Once the best and the worst values of each of the three objectives are identified, the membership functions of the three objectives are constructed as follows

$$\mu_{f_1}(X) = \begin{cases} 0, & \text{if } f_1(X) \geq 427.39 \\ \left[\frac{-f_1(X) + 427.39}{44.02} \right] & \text{if } 383.37 < f_1(X) < 427.39 \\ 1, & \text{if } f_1(X) \leq 383.37 \end{cases} \quad (61)$$

$$\mu_{f_2}(X) = \begin{cases} 0, & \text{if } f_2(X) \geq 2.7485 \\ \left[\frac{-f_2(X) + 2.7485}{0.3893} \right] & \text{if } 2.3592 < f_2(X) < 2.7485 \\ 1, & \text{if } f_2(X) \leq 2.3592 \end{cases} \quad (62)$$

$$\mu_{f_3}(X) = \begin{cases} 0, & \text{if } f_3(X) \geq 1372.83 \\ \left[\frac{-f_3(X) + 1372.83}{100.4} \right] & \text{if } 1272.43 < f_3(X) < 1372.83 \\ 1, & \text{if } f_3(X) \leq 1272.43. \end{cases} \quad (63)$$

The membership functions of the design variables are constructed using the bounds given in Table 6 as

$$\mu_{x_j^u}(X) = \begin{cases} 0, & \text{if } x_j \geq 174.9 \\ 1 - \left[\frac{x_j - 159.0}{15.9} \right] & \text{if } 159.0 < x_j < 174.9, \quad j=2, \dots, 11 \\ 1, & \text{if } x_j \leq 159.0. \end{cases} \quad (64)$$

$$\mu_{x_j^l}(X) = \begin{cases} 1, & \text{if } x_j \geq 70 \\ \left[\frac{x_j - 63}{7} \right] & \text{if } 63 < x_j < 70, \quad j=2, \dots, 11 \\ 0, & \text{if } x_j \leq 63. \end{cases} \quad (65)$$

Membership functions of the remaining design variables are constructed in a similar fashion. Figures 5-7 depict the membership functions corresponding to the upper and lower bounds on design variables x_2 – x_{11} and the objective function f_1 . By permitting a 10% leeway, membership functions of the thirty four behavior constraints present in the problem formulation are also constructed. It may be noted that (for this problem) design variable x_{16} is not truly an independent design variable. Its value is determined by the altitude at the beginning of the fifth segment, and the destination altitude.

Since a design vector with the highest degree of membership to the fuzzy decision set is required, the fuzzy multiobjective optimization problem is formulated as

Find X and λ which

Maximize λ

subject to

$$\lambda \leq \mu_{f_i}(X), \quad i=1,2,3 \quad (66)$$

$$\lambda \leq \mu_{g_j}(X), \quad j=1,2,\dots,34 \quad (67)$$

$$\lambda \leq \mu_{x_j^1}(X), \quad j=1,\dots,15,17,\dots,20 \quad (68)$$

$$\lambda \leq \mu_{x_j^2}(X), \quad j=1,\dots,15,17,\dots,20. \quad (69)$$

The above problem has a total of 21 design variables and 75 constraints. The numerical results obtained by solving this mathematical programming problem are presented in Table 7. The optimum solution yields an overall satisfaction level (λ) of 79.9%. The optimum flight trajectories obtained using single and multiple objective optimization techniques are presented in Figs. 2-4 and 8. It can be seen from Fig. 2 that when the objective is to minimize the fuel cost, the optimum flight parameters entail that the helicopter be flown at altitudes ranging from 5000 to 8000 ft for a good part of the total flight path. However, when the objective is to minimize the flight time or the total cost, the helicopter is flown at altitudes below 4500 ft both during the outbound and return segments of the mission. Also, the rates of climb and descent for the return segment of the mission are higher compared to the corresponding values during the outbound flight. This is due to the fact that some fuel has been consumed during the forward leg and the payload has been delivered.

When all the three objectives are considered simultaneously, it is observed that crisp multiobjective optimization (goal attainment etc.) schemes yield flight parameters which require the helicopter be operated under 5000 ft. When the flight trajectory optimization problem is solved using the techniques of fuzzy optimization, the helicopter altitude during each segment of the flight path exceed the corresponding values given by crisp (single and multiobjective) optimization schemes by as much as 1800 to 4000 ft (Fig. 8).

The fuzzy formulation also yielded the best of values for the three objectives at the optimum solution. In fact, the optimum value of f_3 (total cost) for fuzzy multiobjective formulation with relaxed constraints is *lower* than the value obtained when f_3 is considered alone. All the improvements are possible at the expense of relaxation of the maximum altitude constraint. The maximum altitude attained by the helicopter using fuzzy formulation is 8160 ft compared to a maximum value of 8000 ft for the crisp case. There is no need to change any of the other helicopter parameters such as horsepower required, fuel tank capacity, etc.

4.2 MAIN ROTOR OPTIMIZATION

4.2.1 Introduction

The application of mathematical programming techniques to rotorcraft design problems was first suggested by Stepniewski and Kalmbach (1970). Their paper addressed general concepts in applying multivariable search methods to helicopter design problems. A year later, Bielawa (1971) used linearized rotor dynamic equations to design rotor blades for minimum weight subject to constraints on bending torsion flutter stability and natural frequencies. The problem formulation used a total of five design variables to describe the blade structure. Little was done for about ten years following these two early works, but in the last four to five years there has been a renewed interest in the application of optimal design techniques to rotor blade design problems. Bennet (1983) has studied the effect of blade twist distribution on the input power

required for hover, while keeping airfoil, rotor radius, and tip speed unchanged. The results obtained indicate that optimum twist reduces the hover power by 1.55% compared to linear twist. Bennet (1985) has also investigated into the application of single objective optimization techniques for designing of main rotor blades to meet specific mission goals and constraints. The single rotor analysis program written by Schwartzberg (1977) coupled with two nonlinear programming algorithms was used in the study. Friedmann and Shanthakumaran (1984) have applied structural optimization techniques to rotor blade design problems in order to reduce transmitted vibratory forces. The blade dynamic response and stability analysis was based on a fully coupled, flap-lag torsional analysis. A 15-40% reduction in the amplitude of the vibratory forces was obtained by the authors.

4.2.2 Problem Formulation

A design of the main rotor of a helicopter requires an integration of several analytical disciplines such as aerodynamics, structures, noise, and mission analysis to achieve a viable and an efficient design. An application of optimization techniques to rotor design problems can be broadly divided under three categories: (i) global performance design of rotor, (ii) structural design of blades, and (iii) aerodynamic and acoustic design.

The equations used for blade design in this work are limited to momentum theory considerations to minimize the required input information, procedural detail, and computational complexity without markedly compromising the utility of the solutions. The analysis scheme incorporates mathematical models for hover, vertical flight, and forward flight conditions. Mathematical models for engine performance, fuel consumption, and aircraft group weights are also included in the analysis procedure. The design procedure utilizes the following computation scheme: Initially the engine is sized to meet the most demanding power requirement among the hover, vertical climb, and high speed segments of the mission, including the operation of a multi-engined helicopter with one engine inoperative. The fuel weight and the aircraft component group weights are computed next thus enabling the determination of available payload capacity. A comparison of the available payload with the required payload leads to a new gross weight estimation. The analysis loop is reentered with this new gross weight estimate and the iteration is continued until a gross weight compatible with the payload requirements is determined.

The power required is viewed as comprised of three power absorbing components

$$\text{SHP} = \text{HP}_{\text{MR}} + \text{HP}_{\text{TR}} + \text{HP}_{\text{XM}} \quad (70)$$

where SHP is the required shaft horsepower, HP_{MR} , HP_{TR} , and HP_{XM} are the power required by the main rotor, tail rotor, and the transmission system respectively. The power required by the main and tail rotors are further subdivided into several components. For example, the power required by the main rotor for level forward flight is expressed as the sum of parasite, profile, induced power components, and an additional power required due to compressibility and stall effects. The tail rotor power is taken to be the sum of tail rotor profile and induced power requirements. Power requirements for other flight conditions are computed in a similar fashion. The power lost in the transmission system is expressed as a constant percentage of the power required by the main and tail rotors.

The weight of the helicopter is computed based on a statistical analysis of the weights of 59 different helicopters. The weight breakdown includes the weight of the main rotor, body, propulsion and transmission systems, instrumentation, landing gear, and tail rotor. The equations used for the component weights are derived from a multiple regression analysis of existing helicopters. Using the weight breakdown and cost per pound of various components, the cost of the proposed helicopter is determined. The flyover noise level is expressed as a function of tip speed, Mach number, and gross weight number. The handling qualities of the helicopter are determined by the Lock number and the autorotation parameter. The fuel required for a mission is computed in two different manners. The easiest way is to specify the required payload and range for the mission. The second method of determining fuel consumption involves the definition of the mission at ten individual segments. At each segment it is necessary to specify a time at that segment, atmospheric properties, the velocity, and climb conditions. Upon integrating the fuel flow rate over the duration of the segment, the fuel burned during any mission segment is computed. The second method permits a much more detailed description of the helicopter mission. The cruise speed, dash speed, endurance, as well as hover ceiling are also computed as part of the analysis procedure.

The single-rotor helicopter design and performance estimation program (SSP1), developed by Schwartzberg (1977) has been used in this work to design main rotor blades. The main rotor radius, chord, twist, and tip speed are treated as design variables for this problem. The following inequality constraints are considered in the problem formulation

1. Fuel required \leq Fuel available
2. Required payload \leq Available payload
3. Hover blade loading coefficient \leq specified value
4. Maximum advancing tip Mach number \leq specified value
5. Blade loading in forward flight \leq specified value
6. Hover horsepower \leq specified value
7. Hover horsepower for OEI \leq specified value
8. Horsepower for forward flight OEI \leq specified value
9. Horsepower for maximum speed \leq specified value
10. Horsepower for maximum sustained G level \leq specified value
11. Autorotation index \geq specified value
12. Maximum flyover noise level \leq specified value.

The objective functions considered include the following

1. Minimize gross weight
2. Minimize manufacturing cost
3. Minimize empty weight
4. Minimize mission fuel
5. Maximize endurance limit
6. Maximize dash speed
7. Maximize hover ceiling
8. Minimize noise level.

4.2.3 Numerical Results - Crisp Formulation

This example addressing the design of main rotor blades, for a specified mission, involves eight objective functions. The design data for this example is given in Table 8. The optimum blade parameters obtained by solving eight single objective optimization problems are presented in Table 9, whereas the results obtained using various multiple objective optimization techniques are given in Table 10. The objective function values corresponding to these optimum blade parameters are presented in Tables 11 and 12 respectively. It may be noted from Table 11 that when the main rotor is designed for minimum gross weight, the resulting design also has the minimum total cost and minimum empty weight. The optimum design, however, has a low hover ceiling and a low endurance limit. Attempting to maximize the endurance limit and/or hover ceiling results in a noisy design with high gross weight and a high cost of manufacturing. Due to the conflicting nature of these objectives, single objective optimization techniques are unable to obtain a satisfactory solution. Multiobjective optimization techniques presented here in are able to achieve a compromise between the conflicting pairs of objectives. The optimum solution exhibits good performance with respect to most of the objective functions.

4.2.4 Numerical Results - Fuzzy Formulation

Using the results from single objective optimizations, the matrix [M] is constructed as follows

$$[M] = \begin{bmatrix} 2493 & 340703 & 1076 & 446 & 3.39 & 162.2 & 6040.7 & 89.1 \\ 2493 & 340697 & 1076 & 447 & 3.39 & 162.2 & 6039.7 & 89.1 \\ 2493 & 340747 & 1076 & 447 & 3.39 & 162.2 & 6042.1 & 89.1 \\ 2572 & 371346 & 1167 & 435 & 3.43 & 164.9 & 7169.1 & 89.2 \\ 2764 & 431364 & 1345 & 450 & 3.55 & 158.5 & 8815.4 & 91.3 \\ 2569 & 368691 & 1162 & 436 & 3.38 & 165.7 & 6114.7 & 88.4 \\ 2742 & 427478 & 1333 & 439 & 3.55 & 161.7 & 8933.1 & 90.7 \\ 2588 & 374821 & 1181 & 437 & 3.38 & 165.5 & 6047.4 & 88.4 \end{bmatrix} \quad (71)$$

Once the best and the worst values for each of the eight objectives are identified, the membership functions of the eight objectives are constructed as follows

$$\mu_{f_1}(X) = \begin{cases} 0, & \text{if } f_1(X) \geq 2764 \\ \left[\frac{-f_1(X) + 2764}{271} \right] & \text{if } 2493 < f_1(X) < 2764 \\ 1, & \text{if } f_1(X) \leq 2493 \end{cases} \quad (72)$$

$$\mu_{f_2}(X) = \begin{cases} 0, & \text{if } f_2(X) \geq 431364 \\ \left[\frac{-f_2(X) + 431364}{90667} \right] & \text{if } 340697 < f_2(X) < 431364 \\ 1, & \text{if } f_2(X) \leq 340697 \end{cases} \quad (73)$$

$$\mu_{f_8}(X) = \begin{cases} 0, & \text{if } f_8(X) \geq 91.3 \\ \left[\frac{-f_8(X) + 91.3}{2.9} \right] & \text{if } 88.4 < f_8(X) < 91.3 \\ 1, & \text{if } f_8(X) \leq 88.4. \end{cases} \quad (74)$$

Using the bounds on design variables indicated in Table 13, the membership function of design variable x_1 is constructed as follows:

$$\mu_{x_1^u}(X) = \begin{cases} 0, & \text{if } x_1 \geq 880 \\ 1 - \left[\frac{x_1 - 800}{80} \right] & \text{if } 800 < x_1 < 880 \\ 1, & \text{if } x_1 \leq 800 \end{cases} \quad (75)$$

$$\mu_{x_1^l}(X) = \begin{cases} 1, & \text{if } x_1 \geq 625.0 \\ \left[\frac{x_1 - 562.5}{62.5} \right] & \text{if } 562.5 < x_1 < 625.0 \\ 0, & \text{if } x_1 \leq 562.5 \end{cases} \quad (76)$$

Membership functions of the remaining design variables (x_2 – x_4) are constructed in a similar fashion. By permitting a 10% leeway, membership functions corresponding to the twelve behavior constraints present in the formulation are also constructed. Once the membership functions of all the fuzzy objectives and constraints are determined, the resulting fuzzy optimization problem can be stated as

Find X and λ which

Maximize λ

subject to

$$\lambda \leq \mu_{f_i}(X), \quad i=1,2,\dots,8 \quad (77)$$

$$\lambda \leq \mu_{g_j}(X), \quad j=1,2,\dots,12 \quad (78)$$

$$\lambda \leq \mu_{x_j^l}(X), \quad j=1,\dots,4 \quad (79)$$

$$\lambda \leq \mu_{x_j^u}(X), \quad j=1,\dots,4. \quad (80)$$

The above problem has a total of 5 design variables and 28 inequality constraints. The results obtained by solving this mathematical programming problem are presented in Table 14. The optimum solution corresponds to an overall satisfaction level of 36.1%. It is observed from the results obtained by solving crisp single and multiobjective optimization problems that the design variable x_3 (linear twist of the main rotor blades) is always at its lower bound (-20°) at the

optimum solution. This would seem to indicate that when the lower bound is relaxed in the fuzzy formulation, the linear twist of the main rotor blades may go down even further. This is contrary to what is obtained when the fuzzy optimization problem is solved. The linear twist for the fuzzy optimum design is still close to -20° .

4.3 INTEGRATED DESIGN OF PLANAR MECHANISMS

4.3.1 Introduction

The design of high speed mechanisms requires a simultaneous consideration of both kinematic and dynamic criteria. The kinematic considerations require that the difference between the desired and generated motion be minimized over the entire range of motion, whereas the dynamic considerations entail that the dynamic performance measures of the resulting mechanism be optimized. In the conventional approach to mechanism design, the kinematic criteria are met first. Then, at a later stage, with the link geometry already determined, an improvement of the dynamic characteristics is addressed. This treatment of the dynamic aspects of the problem at a later stage can sometimes seriously limit an improvement of the dynamic performance measures. This paper presents a new multiobjective nonlinear programming formulation which allows both kinematic and dynamic characteristics to simultaneously influence the choice of design parameters.

The application of optimization techniques to the design of planar mechanisms is well known. A number of surveys (Root and Ragsdell 1976) are available which furnish comprehensive reviews of the application of optimization techniques to the design of planar mechanisms. However, despite the importance of both kinematic synthesis and dynamic design in the development of high speed mechanisms, most of the work so far has focussed either only on the optimization of kinematic criteria (Han 1966, Pugh 1984) or on the optimization of dynamic criteria alone (Berkof and Lowen 1971, Rao and Kaplan 1986). Relatively few investigations have addressed the design of mechanisms for a simultaneous optimization of kinematic and dynamic characteristics. Fox and Willmert (1967) first addressed the design of four bar path generating mechanism with constraints on input driving torque and ground bearing forces. Conte et al. (1975) have studied the design of a path generating four bar linkage for three precision points while minimizing the maximum value of shaking force over a cycle. Four kinematic parameters were used as design variables to improve the dynamic characteristics of the resulting mechanism. Kakatsios and Tricamo (1984) have used optimization techniques to design four bar path generating mechanisms with improved kinematic and dynamic characteristics. The problem formulation allowed for a trade off between the kinematic and dynamic criteria. The performance of several dynamic measures, which were imposed as inequality constraints, was improved at the expense of kinematic criteria (structural error). Rigelman and Kramer (1987) have used selective precision synthesis method to design a four bar mechanism for minimum input power while satisfying several kinematic and dynamic constraints.

All the works mentioned above which have considered both kinematic and dynamic aspects of mechanism design, lack one or more of the following: i) Consideration of multiple objectives in the nonlinear programming formulation, ii) Ensuring that the resulting mechanism is free of branch, order, and Grashof defects, iii) Addition of counterweights to moving links to improve

the dynamic performance measures of the resulting mechanism, iv) Incorporating techniques to model vague and imprecise statements in mechanism design problems.

Further, these prior formulations utilizing optimization techniques for mechanism design used only one kinematic or dynamic attribute as an objective function and treated the remaining attributes as inequality constraints. Due to the conflicting and competing nature of these criteria, one can seldom select a single attribute which can be used as an objective function of the mathematical programming formulation. Further, by imposing constraints that the remaining kinematic and dynamic criteria be kept below certain *acceptable levels*, one can end up with an impossible or inferior solution depending upon the choice of *acceptable levels*. Multiobjective optimization techniques tend to overcome these difficulties in an efficient manner.

This work presents a new multiobjective formulation for the design of planar high speed mechanisms which overcomes all the previous shortcomings. The formulation is general enough to facilitate the design of a mechanism for motion, path, and function generation tasks. The kinematic characteristics are influenced by link dimensions and orientations, whereas the dynamic performance is improved by varying link dimensions and orientations, and by the addition of counterweights to all moving links. In addition, it is ensured that the designed mechanism is free of any motion defects, namely Grashof's defect, branch defect, and order defect. Further, to account for the presence of vague and imprecise statements in the problem statement, the tools of fuzzy set theory have been used.

4.3.2 Kinematic Criteria

The primary application of optimization techniques to the design of high speed mechanisms considered in the present work, consists of choosing mechanism and counterweight variables for a planar four bar linkage shown in Fig. 9. Two kinematic criteria are considered in the present work. The first criterion may be given as the location and orientation of a rigid body (motion generation), the coordinates of a tracer point along a prescribed path (path generation), coordinated rotations of input and output links (function generation), or some combination of position and orientation specifications. The second criterion deals with the minimization of the deviation of the transmission angle from its ideal value (90°) over the entire range of motion. The dynamic criteria consist of the minimization of input driving torque, ground bearing forces, and the shaking forces and shaking moments transmitted to the ground link over a cycle. The complete optimization problem for a four bar path generating linkage is developed next.

A four bar mechanism is to be synthesized to generate a given path with coordinated rotation of the input link. Using Fig. 9, the coordinates of the path described by the coupler point P are given as:

$$X_{gi} = X_{O_A} + r_2 \cos(\theta_{2s} + \theta_{2i} + \alpha) + r_5 \cos(\theta_{3i} + \alpha) - r_6 \sin(\theta_{3i} + \alpha) \quad (81)$$

$$Y_{gi} = Y_{O_A} + r_2 \sin(\theta_{2s} + \theta_{2i} + \alpha) + r_5 \sin(\theta_{3i} + \alpha) + r_6 \cos(\theta_{3i} + \alpha) \quad (82)$$

where (X_{O_A}, Y_{O_A}) are the coordinates of the ground pivot O_A , α is the angular orientation of the ground link, r_i ($i=1,2,\dots,6$) are the link lengths, θ_{2s} is the starting position of the input link, and θ_{2i} and θ_{3i} are the angular orientations of links 2 and 3 at the i th design position. Let the corresponding desired values of the path coordinates be given as (X_{di}, Y_{di}) . The first objective

(f_1) considered is the minimization of structural error over the entire range of motion:

$$f_1 = \sum_{i=1}^n \epsilon_i^2 = \sum_{i=1}^n \left[\left(X_{di} - X_{gi} \right)^2 + \left(Y_{di} - Y_{gi} \right)^2 \right] \quad (83)$$

where N denotes the number of design points into which the path is divided. The minimization of f_1 can be achieved by varying the link lengths $r_1 - r_6$ and the ground pivot parameters X_{O_A} , Y_{O_A} , and α .

The second kinematic criterion (f_2) is to minimize the deviation of the transmission angle (γ) from its ideal value (90°) over the entire range of motion:

$$f_2 = \delta = \left[\gamma_{\max} - 90 \right]^2 + \left[\gamma_{\min} - 90 \right]^2 \quad (84)$$

where the minimum and maximum values of γ over a complete cycle are given as:

$$\cos \gamma_{\min} = \frac{r_3^2 + r_4^2 - (r_1 - r_2)^2}{2 r_3 r_4} \quad (85)$$

$$\cos \gamma_{\max} = \frac{r_3^2 + r_4^2 - (r_1 + r_2)^2}{2 r_3 r_4}. \quad (86)$$

The following behavior constraints are imposed on the design problem:

The mechanism should satisfy the loop closure equation at each design position. This is achieved by using an equality constraint of the form

$$2 r_2 r_4 \cos(\theta_2 - \theta_4) - 2 r_1 r_4 \cos \theta_4 + 2 r_1 r_2 \cos \theta_2 + r_3^2 = r_1^2 + r_2^2 + r_4^2 \quad (87)$$

at each design position. In addition, the structural error at each design point is constrained to be less than a specified small quantity Δ , i.e.

$$\epsilon_i \leq \Delta \quad i=1,2,\dots,N. \quad (88)$$

A further design restriction, which assures that the input link be a crank, can be stated as:

$$r_1 + r_2 < r_3 + r_4 \quad (89)$$

$$\left[r_3 - r_4 \right]^2 < \left[r_1 - r_2 \right]^2. \quad (90)$$

In addition, the value of transmission angle (γ) over the entire cycle is constrained as:

$$\frac{\pi}{6} \leq \gamma \leq \frac{5\pi}{6}. \quad (91)$$

Further, to ensure that the resulting mechanism is free of any branch and order defects, a set of four inequality constraints are imposed at each design position. These constraints are developed later.

In the present work, a value of $N=10$ is used and the coordinates of the prescribed path are assumed as follows (Han 1966):

$$X_{di} = 0.4 - \sin 2 \pi (t_i - 0.34) \quad (92)$$

$$Y_{di} = 2.0 - 0.9 \sin 2 \pi (t_i - 0.5) \quad (93)$$

where

$$t_i = \frac{i - 1}{N}. \quad (94)$$

The coordinated input link orientations are determined using

$$\theta_{2i} = 2 \pi t_i. \quad (95)$$

4.3.3 Dynamic Analysis

Two techniques, namely, *kinetostatic* and *time response* approaches, can be used to study the dynamics of mechanisms. In the kinetostatic approach, the motion of the system is completely known, and the purpose of analysis is to determine bearing reactions, shaking forces etc. resulting from that motion, as well as the driving torque required to produce that motion. A solution to this problem can be written as a set of nonlinear algebraic equations. The time response analysis, on the other hand, involves the determination of motion of the mechanism given the actuating force or torque history. A solution to this problem results in a set of nonlinear differential equations which have to be solved numerically. In the present work, the kinetostatic method of dynamic analysis is employed.

The dynamic analysis procedure described is valid for a general four bar linkage shown in Fig. 9. The rigid links are assumed to have a general shape and the revolute joints are considered to be frictionless. Each of the links has a length r_i , $i=1,2,3,4$, and each of the moving links has a mass m_i and a moment of inertia I_i , $i=2,3,4$ with respect to the center of mass which is defined by r_{gi} and ϕ_i as shown in Fig. 9. The equations of equilibrium for each of the three moving links shown in Fig. 10, results in the following system of equations:

$$F_{02x} = F_{23x} - F_{12x} \quad (96)$$

$$F_{02y} = F_{23y} - F_{12y} \quad (97)$$

$$\begin{aligned} T_s + T_{02} - F_{32x} r_2 \sin \theta_2 + F_{32y} r_2 \cos \theta_2 \\ - F_{02x} r_{g2} \sin(\theta_2 + \phi_2) + F_{02y} r_{g2} \cos(\theta_2 + \phi_2) = 0 \end{aligned} \quad (98)$$

$$F_{03x} = F_{34x} - F_{23x} \quad (99)$$

$$F_{03y} = F_{34y} - F_{23y} \quad (100)$$

$$\begin{aligned} T_{03} + F_{34x} r_3 \sin \theta_3 - F_{34y} r_3 \cos \theta_3 \\ - F_{03x} r_{g3} \sin(\theta_3 + \phi_3) + F_{03y} r_{g3} \cos(\theta_3 + \phi_3) = 0 \end{aligned} \quad (101)$$

$$F_{04x} = -F_{34x} - F_{14x} \quad (102)$$

$$F_{04y} = -F_{34y} - F_{14y} \quad (103)$$

$$T_{04} - F_{34x} r_4 \sin \theta_4 + F_{34y} r_4 \cos \theta_4 - F_{04x} r_{g4} \sin(\theta_4 + \phi_4) + F_{04y} r_{g4} \cos(\theta_4 + \phi_4) = 0. \quad (104)$$

Since all inertia forces (F_{oix} , F_{oiy}) and couples (T_{oi}) are known, one can solve this system of nine equations for the x and y components of the four bearing reactions (F_{12} , F_{23} , F_{34} , F_{14}) and the input driving torque (T_s).

The shaking force (SF) is the resultant force on the ground link:

$$SF = F_{21} + F_{41}. \quad (105)$$

Alternately, by using Eqs. (96-104) one gets:

$$SF_x = F_{02x} + F_{03x} + F_{04x} \quad (106)$$

$$SF_y = F_{02y} + F_{03y} + F_{04y}. \quad (107)$$

The shaking moment about an arbitrary point P on the ground link is given as:

$$SM = -T_s - F_{41x} e_1 \sin \psi_1 + F_{41y} e_1 \cos \psi_1 - F_{21x} e_2 \sin \psi_2 + F_{21y} e_2 \cos \psi_2. \quad (108)$$

When P is the midpoint of link $O_A O_B$ (Fig. 10),

$$e_1 = e_2 = \frac{r_1}{2}$$

$$\psi_1 = 0, \quad \psi_2 = 180$$

and the expression for the shaking moment reduces to

$$SM = \frac{r_1}{2} \left[F_{12y} - F_{14y} \right] - T_s. \quad (109)$$

The dynamic analysis is performed at every five degree rotation of the input link. This results in a total of 72 evaluations during each cycle of crank rotation. The ultimate objective is to design a mechanism which requires the minimum driving torque, and transmits minimum forces and moments to the ground. Thus two objective functions are selected as $f_3 = \max F_{12}$ and $f_4 = \max F_{14}$, where $\max F_{12}$ and $\max F_{14}$ are the maximum values of ground bearing forces realized during one input crank revolution. The next dynamic criterion, f_5 is taken as the $\max T_s$, or the maximum value of input driving torque required over a cycle. Finally the last two objective functions f_6 and f_7 , are chosen as the maximum values of shaking force and shaking moment respectively. Thus the optimization problem has a total of seven objective functions.

4.3.4 Counterweighted Linkage

The dynamic performance of the mechanism is also improved by the addition of counterweights to all moving links. The counterweights are restricted as follows:

- i) Each link i may have only one counterweight i .
- ii) Each counterweight is circular and is tangent to the link pivot point.
- iii) Each counterweight has a radius r_{ci} , thickness t_{ci} , and is located at an angle θ_{ci} with respect to link i as shown in Fig. 11.
- iv) All counterweights have the same density ρ .

The counterweight radii, thicknesses, and locations are treated as design variables. The combined mass and inertia properties of the counterweighted link are obtained as follows:

$$\bar{m}_i = m_i + m_{ci} \quad (110)$$

$$(\bar{m}_i \bar{R}_i)^2 = (m_i r_{gi})^2 + (m_{ci} r_{ci})^2 + 2 m_i r_{gi} m_{ci} r_{ci} \cos(\phi_i - \theta_{ci}) \quad (111)$$

and

$$\bar{\theta}_i = \tan^{-1} \frac{m_i r_{gi} \sin \phi_i + m_{ci} r_{ci} \sin \theta_{ci}}{m_i r_{gi} \cos \phi_i + m_{ci} r_{ci} \cos \theta_{ci}}. \quad (112)$$

The counterweight properties are given as follows:

$$m_{ci} = \pi \rho r_{ci}^2 t_{ci} \quad (113)$$

$$I_{ci} = \frac{m_{ci} r_{ci}^2}{2}. \quad (114)$$

Therefore,

$$\bar{I}_i = I_i + I_{ci} + m_{ci} e_i^2 + m_i d_i^2 \quad (115)$$

or,

$$\bar{I}_i = I_i + I_{ci} + m_{ci} r_{ci}^2 + m_i r_{gi}^2 - \bar{m}_i \bar{R}_i^2. \quad (116)$$

The quantities \bar{m}_i and \bar{I}_i are to be used in place of m_i and I_i in Eqs. (96-104).

4.3.5 Development of Branching Constraint

A mechanism is said to suffer from a branch defect when it meets all the design requirements, but has coupler points on both branches of the coupler curve. This requires the mechanism to be disassembled and reassembled at one or more intermediate positions in order to complete the desired motion. A constraint of the form given by Eq. (130) can be used at each design position to avoid this defect. This constraint is developed as follows:

At the reference position (subscript o), one has (Fig. 12):

$$\vec{O_A A} = \vec{k}_0^{(1)} - \vec{m}^{(1)}. \quad (117)$$

The area of triangle A B O_B is given as

$$\Delta = \frac{1}{2} r_3 r_4 \sin \mu_o \quad (118)$$

which can also be written as

$$\Delta = \frac{1}{2} \det \quad (119)$$

where

$$\det = \left[(k_{0x}^{(2)} - k_{0x}^{(1)}) (k_{0y}^{(2)} - m_y^{(2)}) - (k_{0x}^{(2)} - m_x^{(2)}) (k_{0y}^{(2)} - k_{0y}^{(1)}) \right]. \quad (120)$$

Using Eqs. (118-120) one gets

$$\sin \mu_o = \frac{\det}{r_3 r_4}. \quad (121)$$

At the j th position which corresponds to a rotation of link AB by an angle θ_j with respect to the starting position, one gets:

$$\vec{O_A A} = (\vec{k}_0^{(1)} - \vec{r}_0) e^{i\theta_j} + \vec{r}_j - \vec{m}^{(1)} \quad (122)$$

$$\vec{O_B B} = \vec{r}_j + (\vec{k}_0^{(2)} - \vec{r}_0) e^{i\theta_j} - \vec{m}^{(2)}. \quad (123)$$

The area of triangle A B O_B is given as:

$$\Delta = \frac{1}{2} r_3 r_4 \sin \mu_j \quad (124)$$

which yields:

$$\sin \mu_j = \frac{A_j B_j - C_j D_j}{r_3 r_4} \quad (125)$$

where

$$A_j = (k_{0x}^{(2)} - k_{0x}^{(1)}) \cos \theta_j - (k_{0y}^{(2)} - k_{0y}^{(1)}) \sin \theta_j \quad (126)$$

$$B_j = y_j + (k_{0x}^{(2)} - x_0) \sin \theta_j + (k_{0y}^{(2)} - y_0) \cos \theta_j - m_y^{(2)} \quad (127)$$

$$C_j = (k_{0x}^{(2)} - k_{0x}^{(1)}) \sin \theta_j - (k_{0y}^{(2)} - k_{0y}^{(1)}) \cos \theta_j \quad (128)$$

$$D_j = x_j + (k_{0x}^{(2)} - x_0) \cos \theta_j + (k_{0y}^{(2)} - y_0) \sin \theta_j - m_x^{(2)}. \quad (129)$$

A constraint of the form

$$\sin \mu_o \sin \mu_j > 0 \quad (130)$$

at each of the N design positions would ensure that the mechanism does not branch.

4.3.6 Development of Order Defect Constraint

A linkage is said to suffer from an order defect if the designed mechanism is unable to pass through all the design positions in the correct order. For path generation problems, when the input link orientations (ϕ_i) are also treated as design variables, the following three constraints can be added at each design position to ensure that the mechanism is free of any order defect:

$$0 \leq \phi_i \leq 2\pi \quad i=1,2,\dots,N \quad (131)$$

$$\phi_N - \phi_1 \leq 2\pi \quad (132)$$

$$\phi_{i+1} - \phi_i \geq 0 \quad i=1,2,\dots,N-1. \quad (133)$$

4.3.7 Numerical Results - With Linear/Nonlinear Membership Functions

The optimization problem presented has seven objective functions, eighteen design variables and thirty four behavior constraints. In addition, the eighteen design variables are also subject to side constraints limiting their minimum and maximum values. The single objective optimization problems are solved first, and the optimum values of the objective functions are given in Table 15.

Using the results of the single objective optimizations, the procedure outlined under the subsection "Computational Procedure" is followed, and the membership functions for the seven fuzzy goals and the thirty four fuzzy constraints are constructed. This results in a problem with a total of nineteen design variables and forty one constraints (Dhingra and Rao 1989). The results obtained by solving this fuzzy optimization problem are summarized in Table 16. Table 17 presents a comparison between the results obtained when only structural error (f_1) is minimized with those obtained when all seven objectives are considered simultaneously with linear membership functions employed for all the fuzzy goals. It may be noted from Table 17 that when all the seven objectives are considered simultaneously, the dynamic characteristics of the resulting mechanism improve by factors ranging from 3.65 to 13.34 when compared with the mechanism for which only the structural error is optimized. These improvements in dynamic performance measures have been achieved at the expense of structural error (f_1) which has worsened by a factor of 8.22. Keeping in mind that all the remaining kinematic and dynamic performance measures ($f_2 - f_6$) have improved substantially, a structural error of 0.2161 associated with the optimum design is still fairly low from the viewpoint of a mechanism designer.

When nonlinear shapes are employed for the membership functions corresponding to the fuzzy objective functions, the kinematic and dynamic performance measures of the resulting mechanism exhibit similar trends for various values of the membership satiation coefficient. The results obtained using quadratic, exponential ($k < 0$), and logarithmic membership functions are similar, as these three functions model a negative value of $m(X)$. However, the results obtained using these three functions are fairly different from those obtained with membership functions which model constant or increasing marginal membership satiation values. Hence, it is important to accurately assess the nature of the membership functions (e.g., concave, convex, or linear), i.e. the sign of the membership satiation coefficient influences the optimum results significantly.

An insight into the nature of this design problem in conjunction with the optimization results given in Table 15 led to the conclusion that if f_2 (i.e. δ) is dropped from the goal set, the remaining objectives can be improved significantly. This is borne out by the results obtained (Table 18) when only six objectives are considered in the multiobjective problem. It can be observed from Table 17 that, when all the six objectives (f_1 , and $f_3 - f_7$) are considered simultaneously, the dynamic characteristics of the resulting mechanism show an even greater improvement when compared with the linkage for which only the structural error is optimized. The improvement factors for the dynamic performance measures vary from 38.6 to 91.8. Further, the structural error has increased by a much smaller amount (1.19) compared to the case when all seven objectives are considered (8.22). These substantial improvements in both kinematic and dynamic performance measures have been achieved at the expense of the transmission angle objective (f_2) which has worsened by a small (1%) amount with respect to the starting point. Thus, for this planar mechanism design problem, substantially improved kinematic and dynamic characteristics can be obtained when only six objectives are considered.

The results obtained using the multiobjective formulation presented also represent a significant improvement over those obtained by Kakatsios and Tricamo (1984). It can be seen from Table 17 that the improvement factors vary anywhere from 2.59 (for structural error) to 59.85 (for input driving torque) when comparing the Kakatsios and Tricamo results to the multiobjective formulation results using six criteria. This corroborates an observation made earlier that prior formulations which have used only one kinematic or dynamic attribute as an objective function, and treated the remaining attributes as inequality constraints, can lead to inferior solutions. This results from the fact that the conflicting and competing nature of several kinematic and dynamic criteria seldom permits one to select a single attribute which can be used as an objective function in the mathematical programming formulation. It has been demonstrated that proposed multiobjective optimization techniques overcome these difficulties in an efficient manner. Further, the influence of nonlinear shapes modeling various marginally increasing and decreasing membership values has also been examined under the purview of this work. The combined effect of both the membership functions and the fuzzy aggregation operators on the overall design process is a subject of potential future research.

5. CONCLUSIONS

The concept of Pareto-optimal solutions in the context of crisp and fuzzy helicopter design problems is introduced. Several techniques for generating Pareto-optimal solutions are discussed. The effectiveness of multiple objective optimization techniques in the formulation and solution of design problems are demonstrated via an application to helicopter design problems. These techniques are expected to provide a systematic methodology to formulate and solve multiobjective problems in a form directly applicable to engineering design. A comparison is also made between the relative efficiency of various multiobjective techniques. It is found that there is The set of Pareto-optimal solutions generated using the seven different crisp multiple objective optimization techniques presented earlier are, in general, different from each other. This is due to the fact that each methodology has a different underlying preference structure supplied by the decision maker.

Methods such as the global criterion formulation require no articulation of preference structure on part of the DM and can lead to a solution for which a particular objective function value at the optimum solution may be completely unacceptable to the DM. Methods which require an a priori articulation of preference information on part of the DM include the utility function, bounded objective, goal programming, goal attainment, and lexicographic formulations. The major advantage of utility function methods is that if the utility function is correctly assessed and used, it will ensure a most satisfactory solution to the DM. However, the assessment of utility function for even a simple problem is very difficult. The bounded objective, goal programming, goal attainment methods are computationally efficient, but they yield solutions which are fairly sensitive to the goal vector prescribed by the DM. The game theory approach not only provides an optimum design vector, but also yields an optimal set of weights for various objective functions. This in turn leads to a most satisfactory solution with respect to all the competing objectives. However, the computational effort required is the maximum for this approach. Thus, there is no multiple objective optimization technique which can be considered superior to all other techniques. Consequently, one should exercise caution when using a particular technique to solve a class of multiple objective engineering design problems.

The fuzzy optimization techniques presented in this work are expected to be extremely useful during the initial stages of conceptual design of engineering systems where the design goals and constraints have not been clearly identified or stated. These techniques can effectively model the vague and imprecise information present in the objective function and constraints to formulate fuzzy goals and constraints. These models can be used efficiently and effectively for decision making problems in ill-structured situations. Further, since these techniques result in a unified approach to a decision making process in the sense that there is no longer any distinction between the goals and the constraints, they are, in general, able to achieve a superior solution compared to other multiobjective optimization techniques.

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Acknowledgment

The authors would like to thank NASA-Ames Research Center, Moffett Field, CA for supporting this research through NASA-Ames University Consortium Interchange Number NCA2-223 and the NASA collaborators Dr. Hirokazu Miura and Dr. George Kidwell, Jr. of the Advanced Plans and Programs Office, and Dr. Paul Gelhausen of the Powered Lift Flight Investigations Branch for their encouragement, valuable suggestions and contributions to this work.

Table 1 Baseline Mission Profile Parameters.

Mission Requirements

Payload: 2200 lb	Range: 180 n miles
Crew weight: 190 lb	Fuel reserve: 333 lb
Altitude at takeoff: SLS	Altitude of destination: SLS
Maximum flight altitude: 8000 ft	Weight empty: 11500 lb
Maximum rate of climb: 1200 ft/min	Maximum rate of descent: - 600 ft/min
Fuel cost: \$1.25/gal	Maintenance cost: \$360/hr
Maximum internal gross weight: 17500 lb	Fuel capacity: 2958 lb
Hover transmission limit: 2350 hp	Transmission limit: 1950 hp

Table 2 Mission Parameters for Single Objective Optimizations.

Mission Parameters	Starting Vector	Minimize Fuel cost	Minimize Flight time	Minimize Total cost
Air Speed (kn)	117.5	125.2	134.9	136.1
	117.5	103.1	128.5	127.9
	117.5	94.1	125.2	124.1
	117.5	90.1	124.8	122.6
	117.5	90.3	127.7	124.1
	117.5	128.8	151.6	152.0
	117.5	102.0	135.8	134.4
	117.5	88.4	127.9	125.5
	117.5	81.9	124.1	120.6
	117.5	78.8	123.0	118.7
Rate of Climb (ft/min)	0.011	207.7	123.7	134.2
	0.011	91.8	39.3	43.9
	0.011	45.5	12.1	21.5
	0.011	10.4	-1.6	14.0
	0.011	-363.6	-188.5	-230.4
	-0.011	253.4	152.0	169.0
	-0.011	148.8	84.4	94.4
	-0.011	72.7	40.8	48.7
	-0.011	15.2	10.5	20.2
	-0.011	-481.4	-292.9	-337.3
Fuel (lb)	2958.0	2418.8	2658.4	2635.1

Table 3 Mission Parameters for Global, Utility, Lexicographic, Goal Attainment, and Game Theory Formulations.

Mission Parameters	Global Criterion	Utility Function	Lexicographic	Goal Attainment	Game Theory [†]
Air Speed (kn)	137.0	136.3	134.9	136.5	133.0
	118.0	125.6	109.9	120.7	110.0
	108.2	120.4	99.1	115.3	112.2
	103.5	118.1	95.4	118.4	108.8
	104.4	119.4	98.1	122.1	140.7
	146.4	150.6	145.2	142.8	132.4
	117.9	130.3	111.3	123.8	99.3
	102.7	121.0	96.2	116.4	97.2
	94.8	116.3	91.8	115.2	92.1
	92.3	114.8	94.4	117.7	90.3
Rate of Climb (ft/min)	201.2	152.5	258.7	160.8	252.2
	94.6	52.7	113.3	56.8	-19.9
	50.4	26.0	44.8	-23.6	50.7
	16.3	12.3	-1.4	-21.7	-600.0
	-377.7	-259.6	-437.6	-180.5	0.1
	260.8	185.7	322.3	180.5	398.1
	163.3	107.3	169.8	78.6	24.7
	94.3	55.6	56.2	17.8	63.2
	37.5	19.9	-22.7	-25.6	23.6
	-564.1	-378.1	-536.9	-257.1	-550.3
Fuel (lb)	2499.4	2607.3	2474.4	2556.9	2471.2

†

$$\vec{w}^* = [0.583, 0.278, 0.139]^T$$

Table 4 Mission Parameters for Goal Programming and Bounded Objective Formulations.[†]

Mission Parameters	Goal-1	Goal-2	Bound-1	Bound-2	Bound-3
Air Speed (kn)	136.1	136.9	127.1	113.2	115.6
	133.2	114.8	103.6	114.6	113.9
	129.0	104.6	92.7	114.7	109.8
	125.4	99.1	85.9	113.7	107.6
	124.6	97.5	84.3	115.2	109.3
	151.0	147.7	131.9	131.0	131.1
	139.6	130.5	108.9	127.8	124.7
	133.1	122.0	94.3	124.2	118.5
	129.1	98.8	86.4	121.7	114.5
	126.9	90.7	80.5	119.7	113.5
Rate of Climb (ft/min)	89.9	213.6	203.4	93.3	112.5
	43.0	98.0	106.2	34.1	63.6
	35.5	57.2	72.8	18.1	31.4
	25.6	43.8	43.9	15.0	14.1
	-203.8	-425.0	-434.6	-182.9	-248.3
	105.1	148.4	213.3	95.1	116.4
	61.5	90.1	152.0	68.8	79.1
	48.5	230.6	89.2	32.8	48.8
	21.2	91.4	47.2	17.4	22.9
	-239.8	-554.3	-491.6	-227.6	-284.1
Fuel (lb)	2645.8	2510.1	2423.4	2590.5	2555.5

†

Goal-1 Goal programming with $p=1$.
 Goal-2 Goal programming with $p=2$.
 Bound- i Bounded objective formulation with i^{th} goal being optimized.

Table 5 Results for Single and Multiple Objective Optimizations.

Objective Function [†]	Fuel Cost	Flight Time	Total Cost	Function Evals.
Start	437.26	3.00	1517.26	
Single-1	383.37	2.75	1372.83	1444
Single-2	427.39	2.37	1279.70	1496
Single-3	423.13	2.36	1272.43	1327
Global	398.18	2.44	1276.72	2105
Utility	418.01	2.37	1272.59	1529
Lexico	393.59	2.46	1278.70	4111
Goal-1	425.07	2.38	1281.68	1435
Goal-2	400.15	2.46	1285.50	1984
Bound-1	384.21	2.72	1363.75	1906
Bound-2	414.91	2.63	1362.17	810
Bound-3	408.50	2.63	1354.22	658
Goal-At	408.75	2.47	1298.03	2022
Game-Th	392.99	2.51	1294.92	35825

†

Single-1	Minimize fuel cost.
Single-2	Minimize flight time.
Single-3	Minimize total cost.
Global	Global criterion formulation.
Utility	Utility function formulation.
Lexico	Lexicographic method with order 1-2-3.
Goal-i	Goal programming with $p=i$.
Bound-i	Bounded objective formulation with i^{th} goal being optimized.
Goal-At	Goal attainment formulation.
Game-Th	Game theory formulation.

Table 6 Bounds on Design Variables for Flight Profile Optimization.

Design Variable	Crisp Lower Bound	Crisp Upper Bound	Fuzzy Lower Bound	Fuzzy Upper Bound
x_1	10.0	2957.7	9.0	3253.5
$x_2 - x_{11}$	70.0	159.0	63.0	174.9
$x_{12} - x_{15}$	-600.0	1200.0	-660.0	1320.0
x_{16}	99.9	100.1	99.9	100.1
$x_{17} - x_{20}$	-600.0	1200.0	-660.0	1320.0

Table 7 Mission Parameters for Fuzzy Optimization.

Design Variables

Air Speeds: 140.3, 107.4, 95.3, 92.5, 99.0,
(kn) 148.0, 109.8, 93.5, 90.8, 92.8

Rates of Climb: 297.8, 129.1, 38.2, -43.5, -434.8,
(ft/sec) 352.3, 182.0, 36.7, -12.4, -564.1

Initial Fuel = 2418.4 lbs $\eta = 0.7987$

Objective Functions

Fuel Cost = 392.29, Flight Time = 2.438 hrs, Total Cost = 1269.81

Table 8 Design Parameters for a Typical Small Helicopter.

Baseline Airframe: OH-58A **Baseline Engine:** T63
Required Payload: 970 lb **Required Range:** 300 n miles at SLS

Design Variables and bounds

	Minimum Value	Initial Value	Maximum Value
Tipspeed (ft/sec)	625.0	642.14	800.0
Radius (ft)	10.0	14.37	18.0
Chord (ft)	0.5	0.59	2.0
Twist (deg)	-20.0	-16.84	0.0

Engine Sizing Points

- | | |
|---------------------------------------|----------------|
| 1. Hover OGE at 6000' / 37.6°F | HP Avail = 350 |
| 2. Level Flight Speed = 132 kn at SLS | HP Avail = 302 |

Design Constraints

1. Engine power required for engine sizing point #1
2. Engine power required for engine sizing point #2
3. Max advancing tip mach number < 0.95
4. Hover blade loading coefficient $2 C_T < 0.18$
5. Forward flight blade loading coefficient < 0.50 - 0.46 μ
6. Minimum $t / K > 0.5$
7. Maximum noise level < 93 dB

Table 9 Design Variables for Single Objective Optimizations.

Objective function	MR Radius (ft)	Chord (ft)	Twist (deg)	Tip Speed (ft/sec)	Function Evals.
Starting Point	14.37	0.59	-16.84	642.14	
Gross Weight	11.90	0.50	-20.00	730.66	459
Total Cost	11.90	0.50	-19.99	730.74	464
Empty Weight	11.91	0.50	-20.00	730.80	410
Fuel Weight	13.31	0.53	-20.00	668.53	357
Endurance	16.00	0.50	-20.00	690.18	261
Dash Speed	12.49	0.59	-20.00	656.41	391
Hover Ceiling	16.00	0.50	-20.00	654.73	230
Noise Level	12.58	0.61	-20.00	647.68	363

Table 10 Design Variables for Multiple Objective Optimizations.

Solution Technique [†]	MR Radius (ft)	Chord (ft)	Twist (deg)	Tip Speed (ft/sec)
Global	13.59	0.5	-19.99	679.10
Utility	12.86	0.50	-20.00	689.52
Goal-1	12.87	0.50	-20.00	689.48
Goal-2	14.09	0.50	-20.00	672.55
Lexico	12.21	0.56	-20.00	681.37
Bound-1	12.19	0.50	-20.00	725.06
Bound-2	12.18	0.50	-20.00	725.10
Bound-3	12.18	0.50	-20.00	725.11
Bound-4	13.46	0.52	-20.00	669.56
Bound-5	14.94	0.53	-19.22	647.88
Bound-6	14.35	0.59	-16.65	638.32
Bound-7	14.63	0.51	-17.25	660.05
Bound-8	13.24	0.65	-20.00	625.04
Goal-At	12.28	0.50	-20.00	712.85
Game-Th	13.38	0.50	-19.92	682.10

†

Global	Global criterion formulation.
Utility	Utility function formulation.
Goal-i	Goal programming with $p=i$.
Lexico	Lexicographic method with order 1-2-3-4-5-6-7-8.
Bound-i	Bounded objective formulation with i^{th} objective being optimized.
Goal-At	Goal attainment formulation.
Game-Th	Game theory formulation.

Table 11 Objective Function Values for Single Objective Optimization.

Objective function	Gross Weight (lb)	Total Cost	Empty Weight (lb)	Fuel Weight (lb)	Endur. Limit (hr)	Dash Speed (kn)	Hover Ceiling (ft)	Noise Level (dB)
Starting Point	2696	410 482	1285	441	3.47	162.8	7363.3	89.6
Min Gross Wt	2493	340 703	1076	446	3.39	162.2	6040.7	89.1
Min Total Cost	2493	340 697	1076	447	3.39	162.2	6039.7	89.1
Min Empty Wt	2493	340 747	1076	447	3.39	162.2	6042.1	89.1
Min Fuel Wt	2572	371 346	1167	435	3.43	164.9	7169.1	89.2
Max Endur. Limit	2764	431 364	1345	450	3.55	158.5	8815.4	91.3
Max Dash Speed	2569	368 691	1162	436	3.38	165.7	6114.7	88.4
Max Hover Ceiling	2742	427 478	1333	439	3.55	161.7	8933.1	90.7
Min Noise Level	2588	374 821	1181	437	3.38	165.5	6047.4	88.4

Table 12 Objective Function Values for Multiple Objective Optimization.

Solution Technique	Gross Weight (lb)	Total Cost	Empty Weight (lb)	Fuel Weight (lb)	Endur. Limit (hr)	Dash Speed (kn)	Hover Ceiling (ft)	Noise Level (dB)
Global	2572	371 720	1167	435	3.45	164.3	7590.9	89.5
Utility	2527	356 329	1121	436	3.42	164.7	7032.5	89.2
Goal-1	2527	356 344	1121	436	3.42	164.7	7033.2	89.2
Goal-2	2605	382 672	1199	436	3.47	163.9	7927.9	89.8
Lexico	2536	357 542	1128	438	3.37	165.2	6051.5	88.6
Bound-1	2506	345 618	1090	445	3.40	162.2	6330.6	89.2
Bound-2	2506	345 606	1090	445	3.40	162.2	6329.7	89.2
Bound-3	2506	345 595	1090	445	3.40	162.2	6329.1	89.2
Bound-4	2577	373 248	1172	435	3.44	164.7	7323.4	89.3
Bound-5	2688	409 798	1281	437	3.50	163.2	8157.9	90.0
Bound-6	2698	411 260	1288	440	3.47	163.1	7369.5	89.5
Bound-7	2660	400 488	1254	437	3.49	163.5	8144.5	90.0
Bound-8	2659	397 657	1250	439	3.42	164.3	6202.3	88.5
Goal-At	2504	346 509	1093	441	3.40	163.4	6472.7	89.1
Game-Th [†]	2559	367 215	1154	436	3.44	164.4	7438.2	89.4

†

$\vec{w}^* = [0.1132, 0.112, 0.1188, 0.1464, 0.1288, 0.1211, 0.1361, 0.1236]$

Table 13 Bounds on the Design Variables for Main Rotor Design.

Design Variable	Crisp Lower Bound	Crisp Upper Bound	Fuzzy Lower Bound	Fuzzy Upper Bound
x_1	625.0	800.0	562.5	880.0
x_2	10.0	16.0	9.0	17.6
x_3	0.5	2.0	0.45	2.2
x_4	-20.0	0.0	-22.0	0.0

Table 14 Main Rotor Parameters for Fuzzy Optimization.

Design Variables

MR Radius = 13.94 ft MR Chord = 0.52 ft
MR Twist = -19.98 deg Tip Speed = 656.12 ft/sec
 $\lambda^* = 0.361$

Objective Functions

Gross Weight = 2600 lbs Total Cost = 381 654
Empty Weight = 1197 lbs Fuel Weight = 433 lbs
Endurance = 3.46 hrs Dash Speed = 165.0 kn
Hover Ceiling = 7720 ft Noise Level = 89.4dB

Table15 Objective function values for single objective optimizations.

Objective Function	Starting Vector	Min. ϵ	Min. δ	Min. F_{12}	Min. F_{14}	Min. T_s	Min. SF	Min. SM
$\epsilon \times 10^2$	55.242	2.63 [*]	65.88	65.81	37.64	65.83	17.27	32.4
δ	1301.4	1379.6	266.8 [*]	699.3	1421.3	946.7	1274.3	1371.2
F_{12}	15.988	15.38	16.964	0.0944 [*]	22.791 [*]	22.198	0.9837	0.1548
F_{14}	8.5648	7.524	2.8501	0.1469	0.1051 [*]	0.4149	0.9663	0.1191
T_s	4.6061	3.654	0.8806	0.0464	0.1413	0.0401 [*]	0.2561 [*]	0.0635
SF	18.884	19.34	17.452	0.1698	22.882	22.396	0.0561 [*]	0.1648 [*]
SM	25.479	23.61	45.228	0.1489	41.098	35.933	1.5398	0.1235 [*]

Legend

- Min. ϵ Minimize Structural error.
- Min. δ Minimize Transmission angle deviation.
- Min. F_{12} Minimize maximum value of F_{12} .
- Min. F_{14} Minimize maximum value of F_{14} .
- Min. T_s Minimize maximum value of driving torque.
- Min. SF Minimize maximum value of shaking force.
- Min. SM Minimize maximum value of shaking moment.
- * Optimal single objective function values.

Table 16 Results obtained with seven objectives considered simultaneously.*

Attribute	Linear	Hyperbolic	Expon. ($k=1.0$)	Quadratic	Logarithm	Expon. ($k=-1.0$)
$\epsilon \times 10^2$	21.61	27.13	19.25	22.57	24.06	23.77
δ	612.6	708.8	568.8	630.9	657.9	652.6
F_{12}	2.528	2.489	2.3598	3.18	3.058	2.399
F_{14}	2.059	1.291	1.699	2.117	2.046	2.291
T_s	0.274	0.552	0.281	0.310	0.258	0.286
SF	3.629	2.677	3.12	4.209	4.141	3.742
SM	4.716	4.324	4.872	5.837	5.462	4.159
λ	0.7027	0.7955	0.6362	0.8285	0.7603	0.7691

Legend

- ϵ Structural error.
- δ Transmission angle deviation.
- F_{12} Maximum value of F_{12} .
- F_{14} Maximum value of F_{14} .
- T_s Maximum value of driving torque.
- SF Maximum value of shaking force.
- SM Maximum value of shaking moment.
- λ Overall satisfaction level.
- * All membership functions are of the same form for a given column, i.e., all seven membership functions are either linear, exponential, etc. No mixtures have been considered.

Table 17 Comparison of single versus multiple objective optimization results. *

Attribute (1)	Structural Error (2)	Seven Objectives (3)	Impr. Factor (4)	Six Objectives (5)	Impr. Factor (6)	Kakatsios Tricamo (7)	Impr. Factor (8) ⁺
$\epsilon \times 10^2$	2.63	21.61	-8.22	3.132	-1.19	8.1	2.59
δ	1379.6	612.6	2.25	1392.9	-1.01		
F_{12}	15.38	2.528	6.08	0.1687	91.17	9.24	54.77
F_{14}	7.524	2.059	3.65	0.1949	38.61	3.61	18.52
T_s	3.654	0.274	13.34	0.0777	47.03	4.65	59.85
SF	19.34	3.629	5.33	0.2106	91.83	9.36	44.44
SM	23.61	4.716	5.01	0.2781	84.89	12.93	46.49

*

Structural Error Only structural error (f_1) is minimized.
 Seven Objectives All seven objectives considered simultaneously (linear membership functions).
 Impr. Factor Improvement over the corresponding values in Column 2. A negative sign indicates a relative worsening (col.(4)=col.(2)/col.(3), col.(6)=col.(2)/col.(5)).
 Six Objectives Only six objectives are considered simultaneously (linear membership functions).
 Kakatsios and Results from Table 3, Run 5 in Ref. [10].
 Tricamo column(8)=col.(7)/col.(5).
 +

Table 18 Results obtained with six objectives considered simultaneously.

Attribute	Linear	Hyperbolic	Expon. (k=1.0)	Quadratic	Logarithm	Expon. (k=-1.0)
$\epsilon \times 10^2$	3.132	4.121	3.111	2.847	2.899	3.462
δ	1392.9	1396.8	1731.4	1567.7	1439.2	2187.1
F ₁₂	0.1687	0.2634	0.1422	0.1882	0.2094	0.1664
F ₁₄	0.1949	0.2982	0.1735	0.2044	0.2071	0.192
T ₃	0.0777	0.0759	0.0726	0.0801	0.0737	0.0706
SF	0.2106	0.2865	0.1986	0.1888	0.1806	0.1954
SM	0.2781	0.4662	0.2126	0.2341	0.3029	0.2142
λ	0.989	0.9968	0.9865	0.9989	0.9964	0.9939

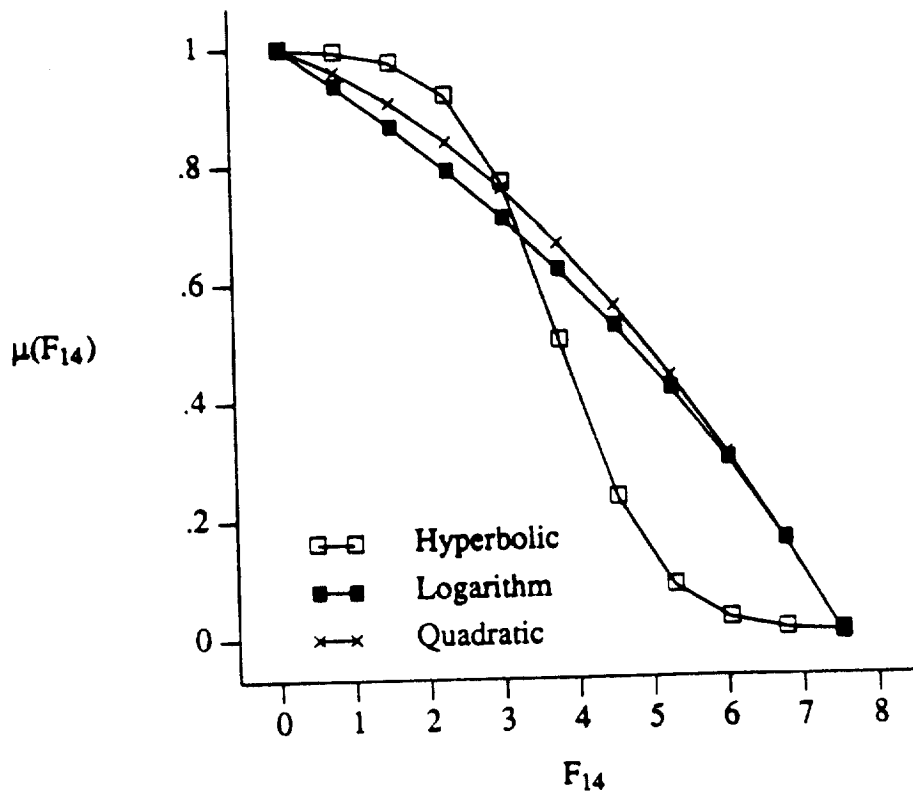
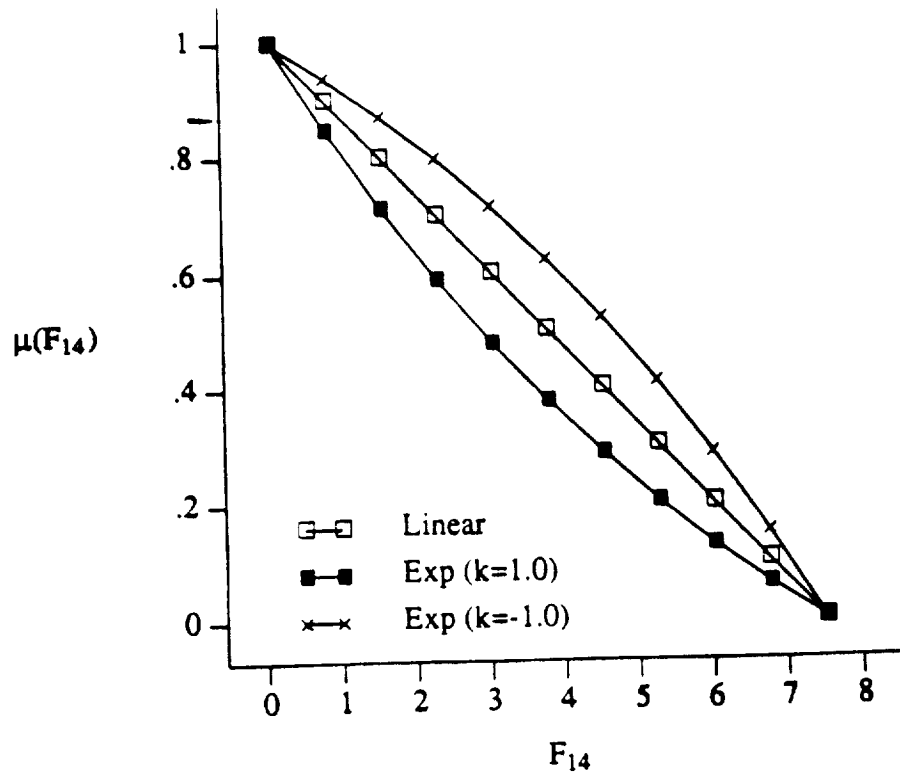


Fig. 1 Nonlinear membership functions for fuzzy goals.

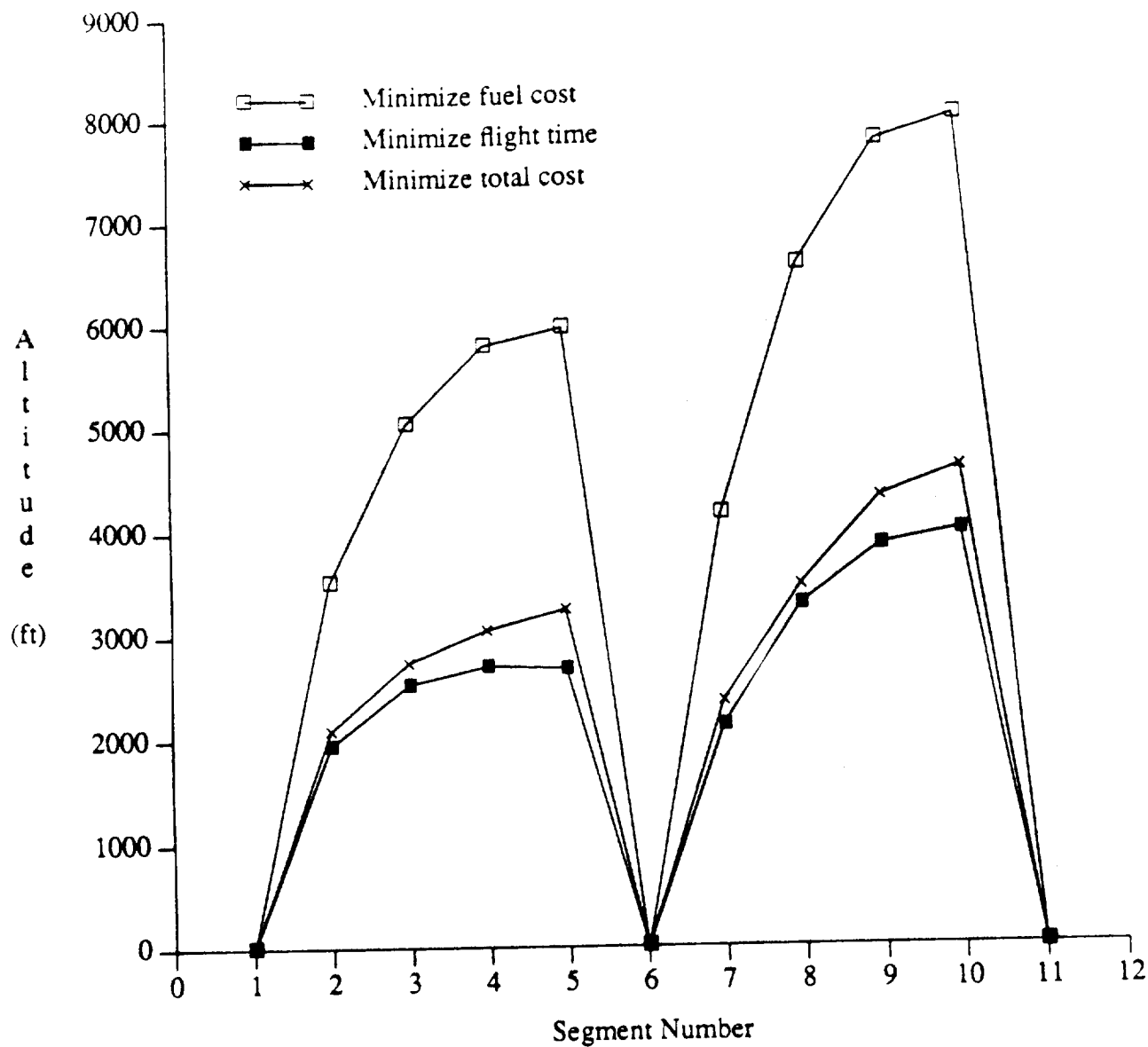


Fig. 2 Flight trajectories for single objective optimizations.

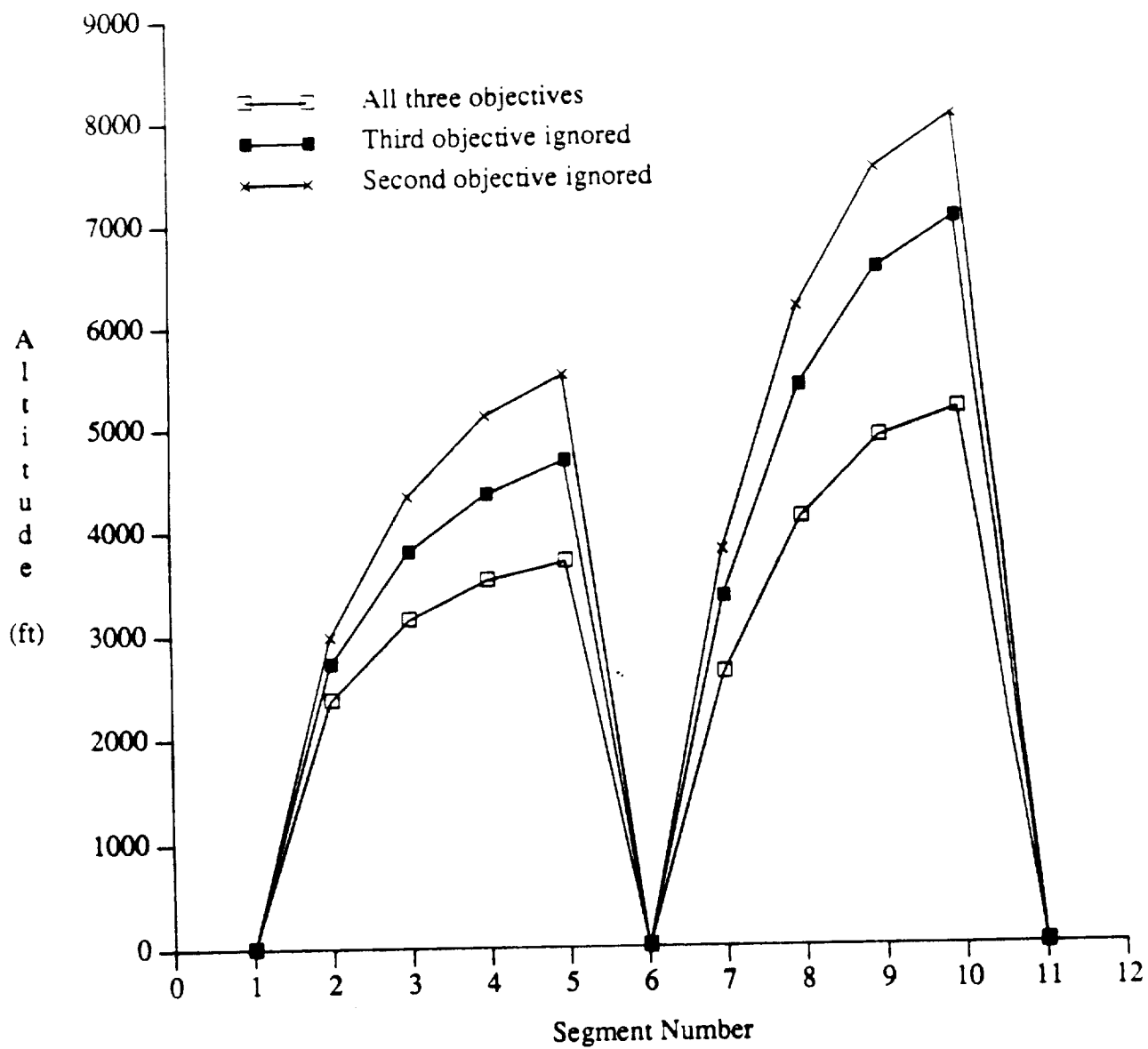


Fig.3 Optimum flight trajectories for Utility function formulation.

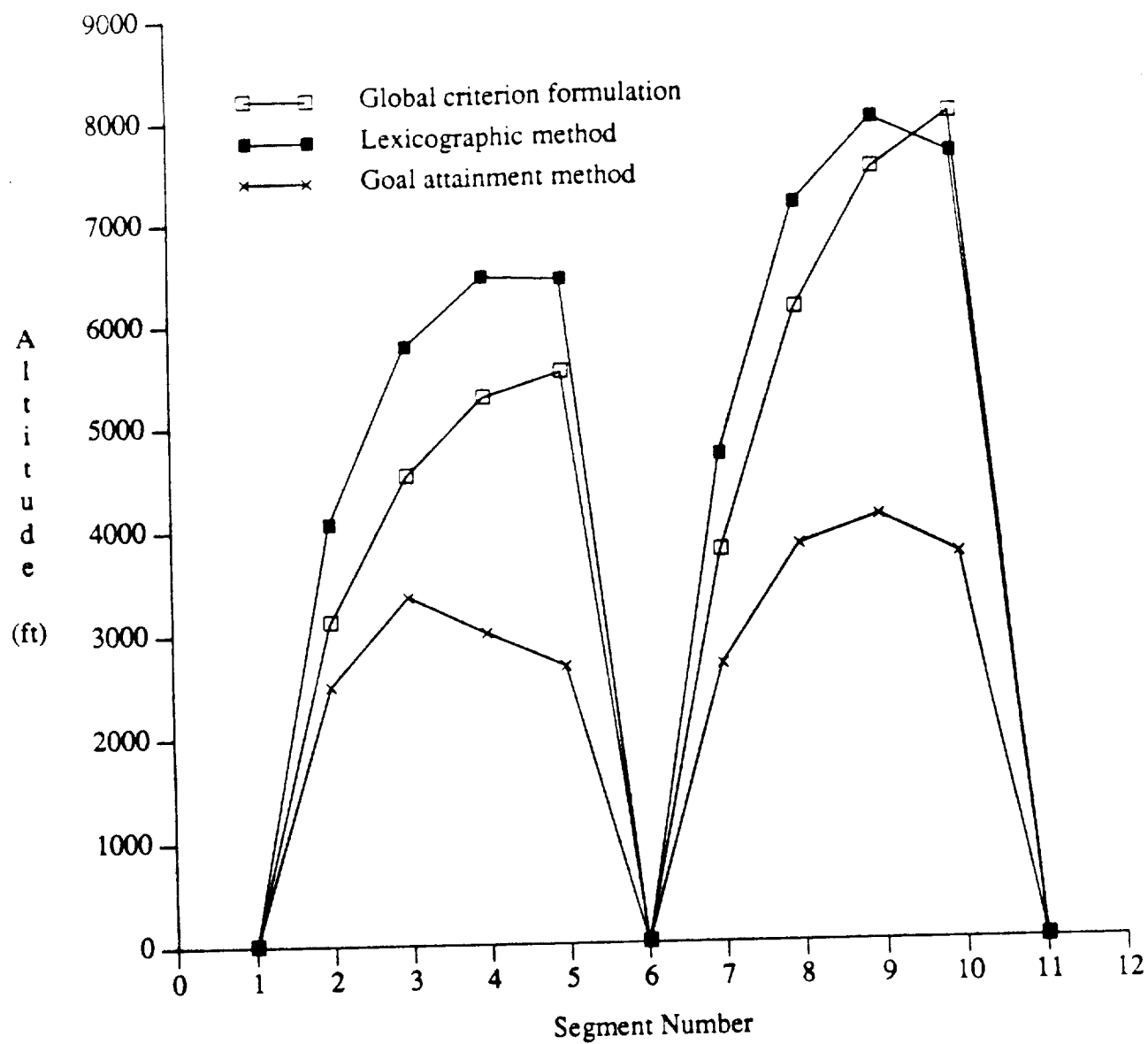


Fig. 4 Flight paths for Global, Goal attainment, and Lexicographic methods.

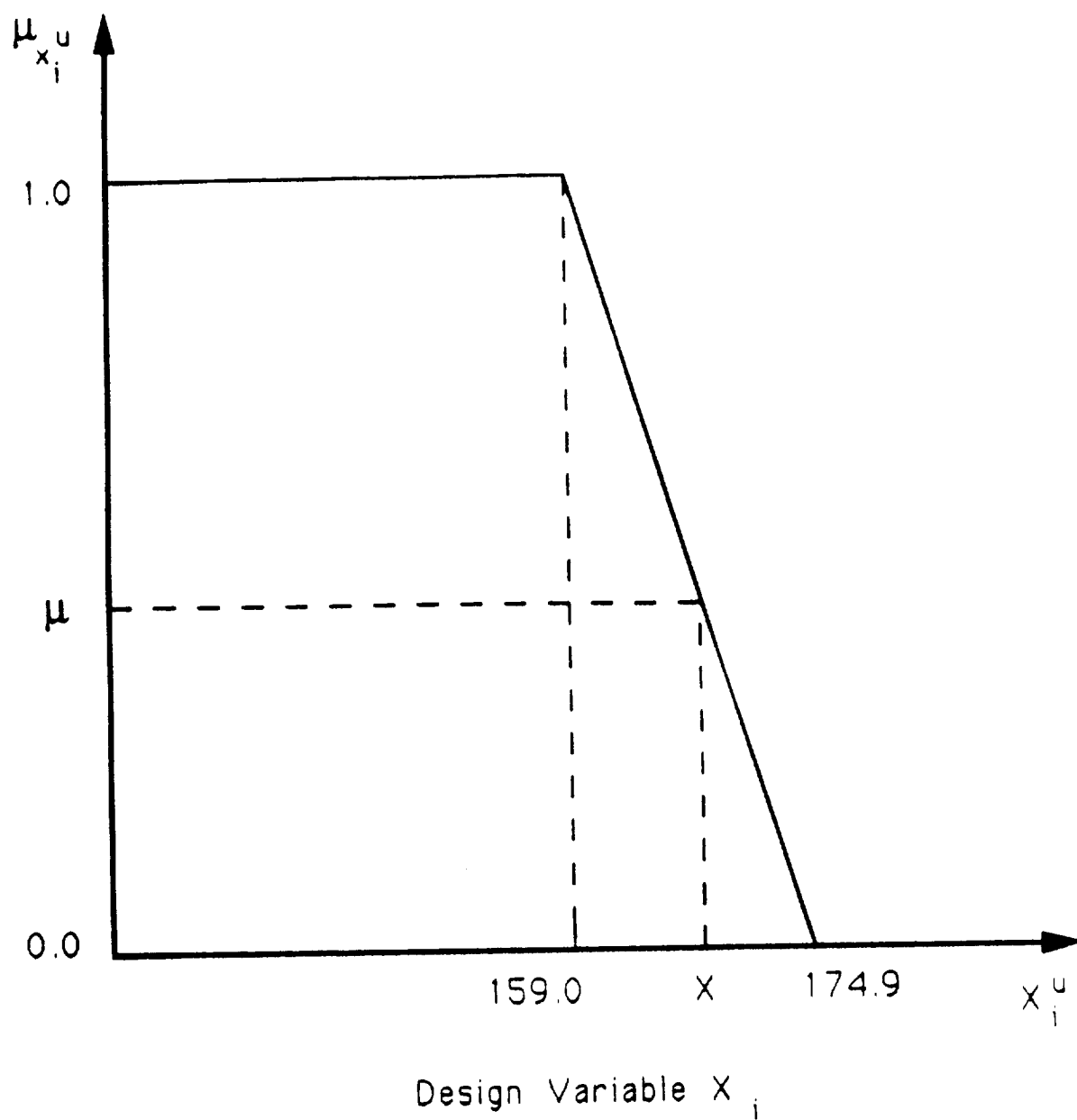


Fig. 5 Membership function for upper bound on the design variable x_i .

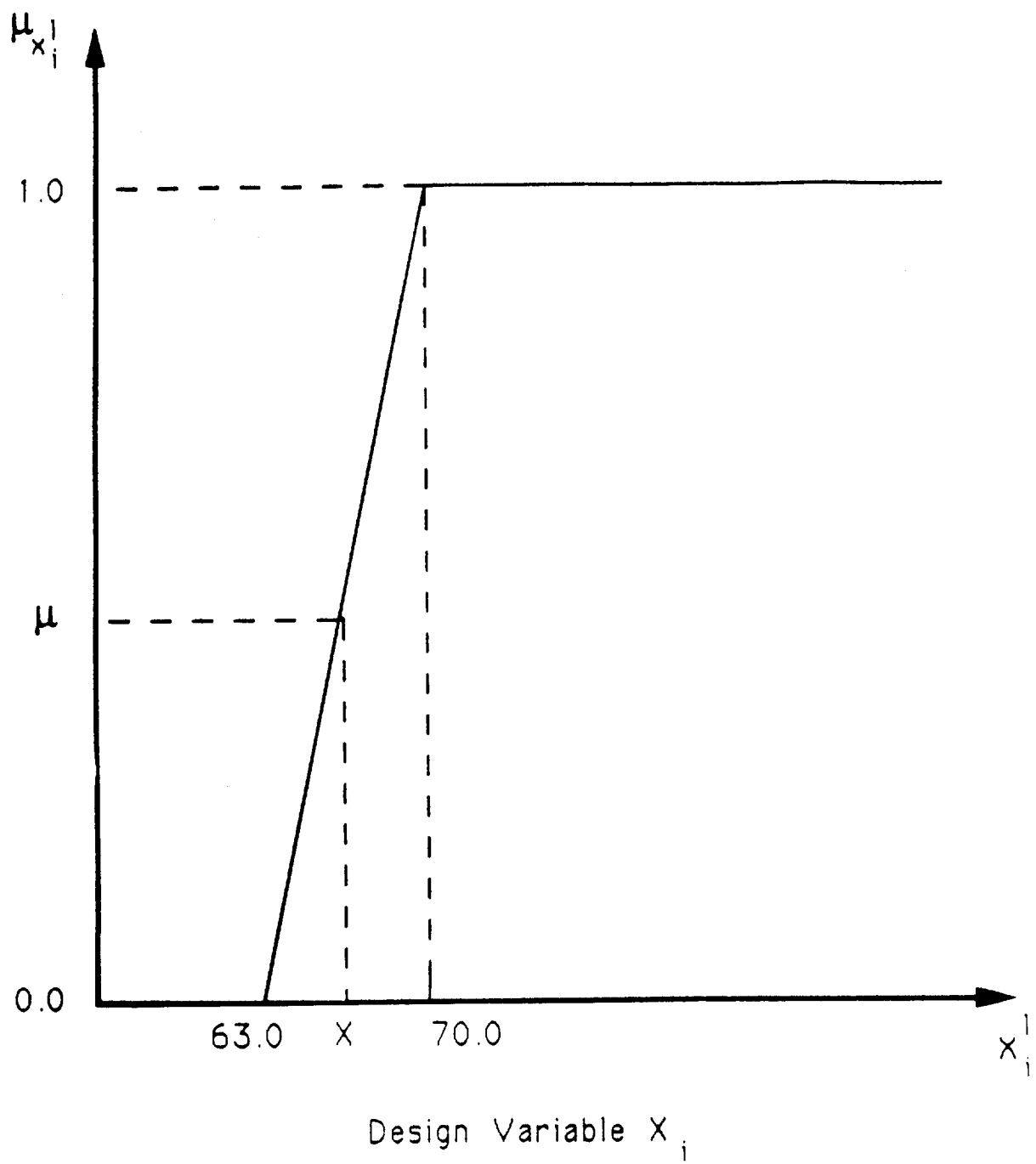


Fig.6 Membership function for lower bound on the design variable x_i .

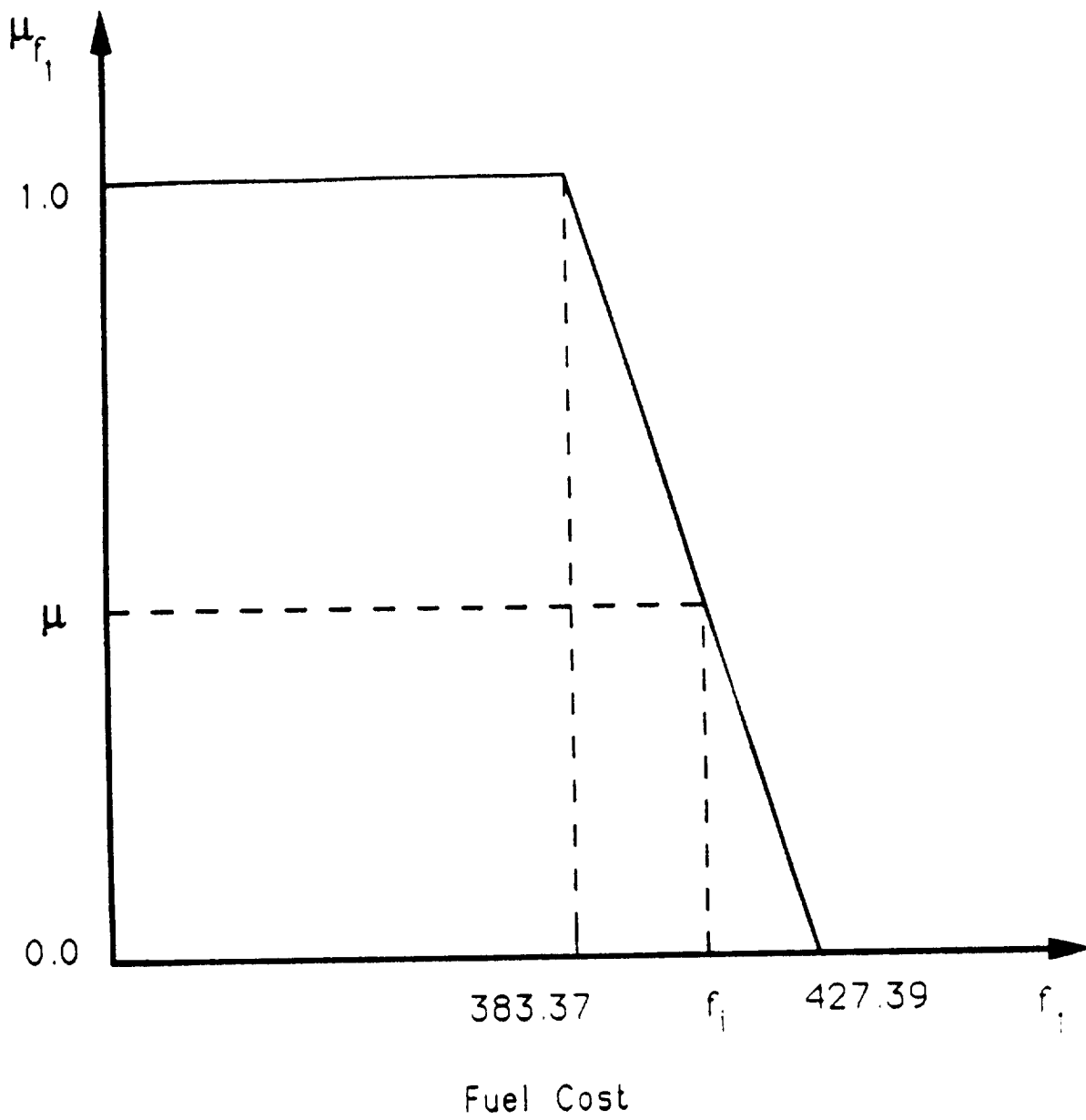


Fig. 7 Membership function for the first objective, f_1 .

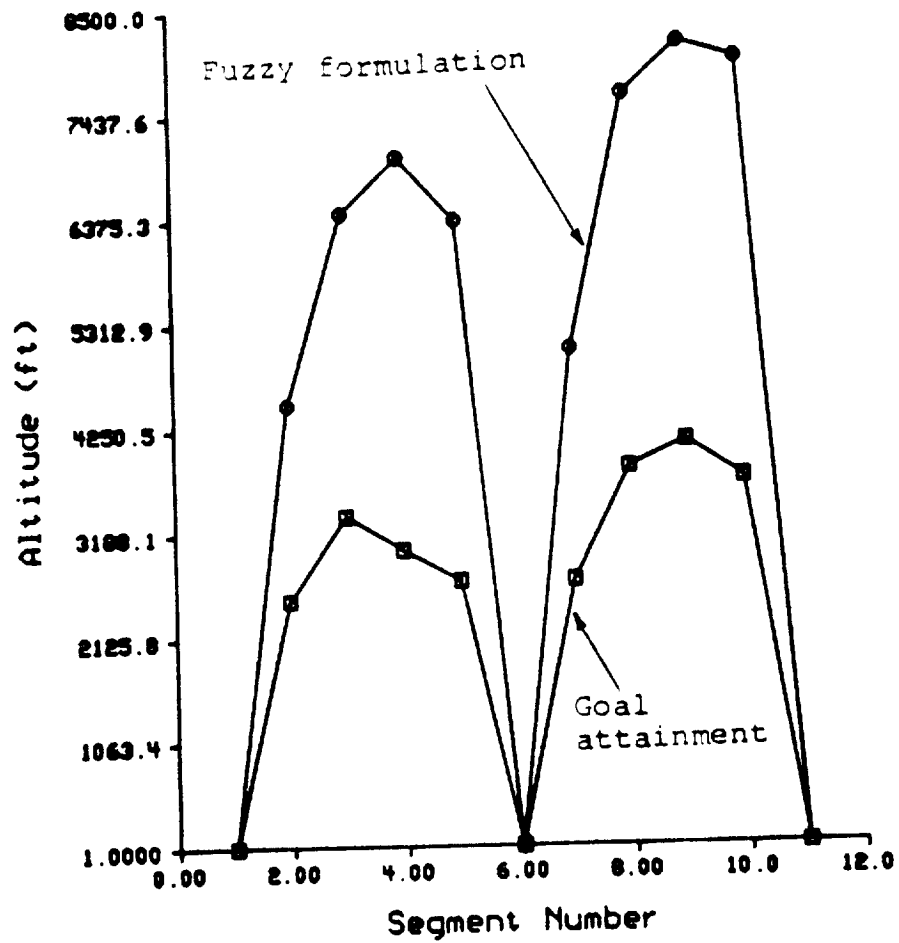


Fig.8 Flight trajectories for goal attainment and fuzzy formulations.

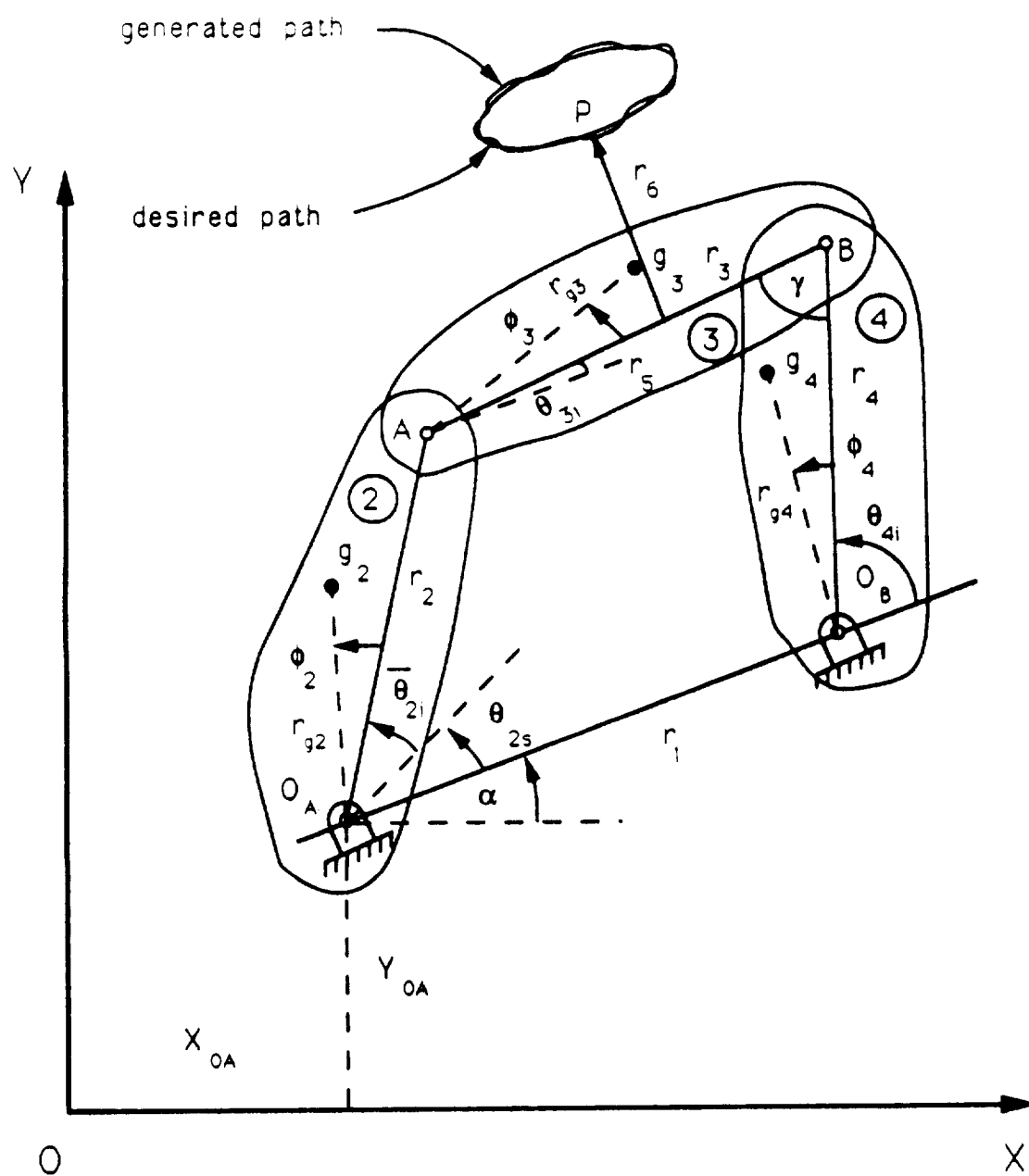
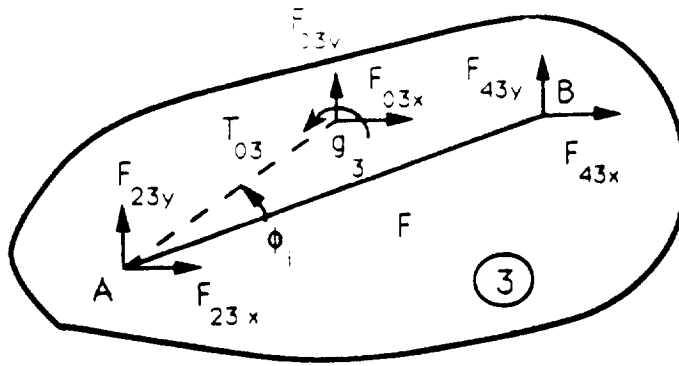
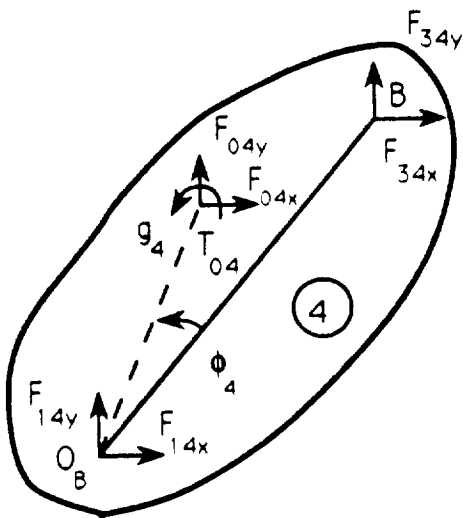


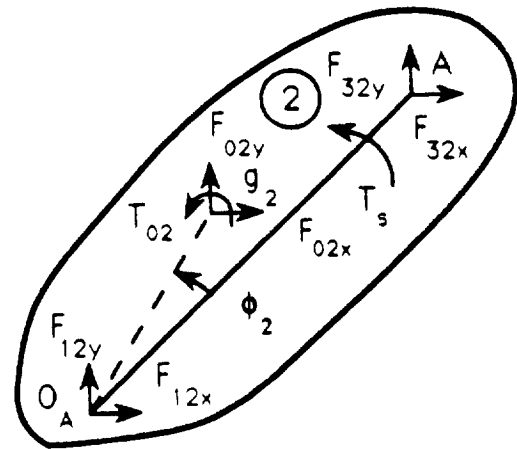
Fig.9 A path generating planar four bar mechanism.



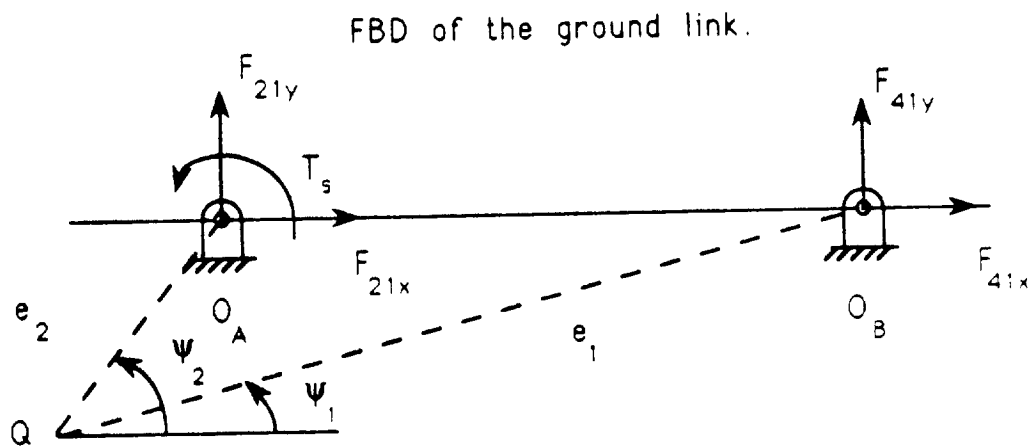
FBD of link 3.



FBD of link 4.



FBD of link 2.



FBD of the ground link.

Fig.10 Free body diagrams (FBDs) used in the dynamic analysis.

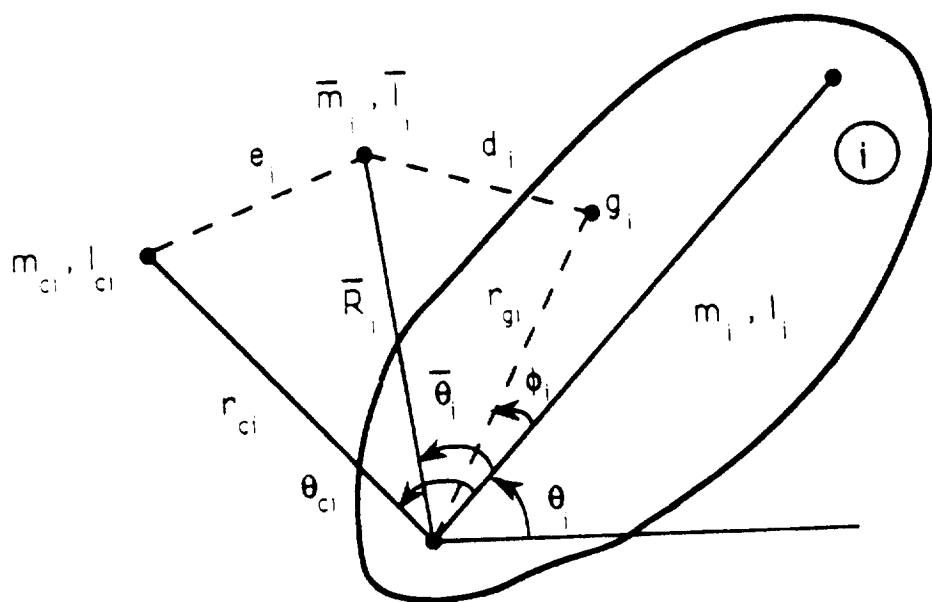


Fig. 11A counterweighted link.

Report Documentation Page

1. Report No. NASA CR-177573		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle Applications of Fuzzy Theories to Multi-Objective System Optimization				5. Report Date January 1991	
				6. Performing Organization Code	
7. Author(s) S. S. Rao and A. K. Dhingra				8. Performing Organization Report No. A-91029	
				10. Work Unit No.	
9. Performing Organization Name and Address School of Mechanical Engineering Purdue University West Lafayette, Indiana 47907				11. Contract or Grant No. NCA2-223	
				13. Type of Report and Period Covered Contractor Report	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546-0001				14. Sponsoring Agency Code	
15. Supplementary Notes Point of Contact: H. Miura, Ames Research Center, MS 237-11, Moffett Field, CA 94035-1000 (415) 604-5888 or FTS 464-5888					
16. Abstract Most of the computer aided design techniques developed so far deal with the optimization of a single objective function over the feasible design space. However, there often exist several engineering design problems which require a simultaneous consideration of several objective functions. This work presents several techniques of multiobjective optimization. In addition, a new formulation, based on fuzzy theories, is also introduced for the solution of multiobjective system optimization problems. The fuzzy formulation is useful in dealing with systems which are described imprecisely using fuzzy terms such as, "sufficiently large," "very strong," or "satisfactory." The proposed theory translates the imprecise linguistic statements and multiple objectives into equivalent crisp mathematical statements using fuzzy logic. The effectiveness of all the methodologies and theories presented is illustrated by formulating and solving two different engineering design problems. The first one involves the flight trajectory optimization and the main rotor design of helicopters. The second one is concerned with the integrated kinematic-dynamic synthesis of planar mechanisms. The use and effectiveness of nonlinear membership functions in fuzzy formulation is also demonstrated. The numerical results indicate that the fuzzy formulation could yield results which are qualitatively different from those provided by the crisp formulation. It is felt that the fuzzy formulation will handle real life design problems on a more rational basis.					
17. Key Words (Suggested by Author(s)) Multi-objective optimization Fuzzy theory Design optimization				18. Distribution Statement Unclassified-Unlimited Subject Category - 05	
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No. of Pages 76	
				22. Price A05	